Laws of Exponents, Logarithms, Algebraic Rules

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NOTE: Not a comprehensive list of algebraic properties/tricks. These are important ones to focus on for this class. Full list of algebraic/trig/calculus notes sheets: https://tutorial.math.lamar.edu/Extras/CheatSheets_Tables.aspx#AlgSheet

- **laws of exponents:**
  - x & y are variables, n & m are numbers

1. $x^n x^m = x^{n+m}$
2. $(x^n)^m = x^{n \cdot m}$
3. $(xy)^n = x^n y^n$
4. $x^{-n} = \dfrac{1}{x^n}$
5. $\left(\dfrac{x}{y}\right)^{-n} = \left(\dfrac{y}{x}\right)^n = \dfrac{y^n}{x^n}$
6. $\dfrac{x^n}{x^m} = x^{n-m}$
7. $x^0 = 1$, $x \neq 0$
8. $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$
9. $x^{\frac{n}{m}} = \left(x^{\frac{1}{m}}\right)^n = \left(x^n\right)^{\frac{1}{m}}$

- **laws of radicals:**
  - $\sqrt[n]{x} = x^{1/n}$
  - $\sqrt[m]{\sqrt[n]{x}} = \sqrt[nm]{x}$
  - $\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$
  - $\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$

- **where we would use it:**
  - * manipulating series/sequences to put in a recognizeable form (ex: geometric)

ex: $b_n = \dfrac{5^{n+1}}{7^n}$. is this sequence convergent or divergent:

rewrite using exponent properties:

$$b_n = \frac{5^n \cdot \underbrace{5}_{\text{constant}}}{7^n} = \underbrace{5}_{a}\underbrace{\left(\frac{5}{7}\right)^n}_{r} \Rightarrow \text{now in geometric form}$$

$a = 5$

$r = 5/7 \rightarrow$ since $r < 1$, converges

- **laws of logarithms:**
  - recall... $y = \log_b x \Rightarrow x = b^y$
  - 2 common logs:
    - $\ln(x) = \log_e x \rightarrow$ natural log
    - $\log(x) = \log_{10} x \rightarrow$ common log

- **properties: (note: $\log_b x$ is only valid for $x > 0$)**
  - $\log_b b = 1$
  - $\log_b b^x = x$
  - $\log_b (x^r) = r \log_b x$
  - $\log_b (xy) = \log_b x + \log_b y$
  - $\log_b \left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

- helpful algebraic tricks for series
  - **"multiplying by 1"**

ex: limit comparison test:

1. $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{1}{\sqrt{n}+1}}_{a_n}$ $\quad b_n = \dfrac{1}{\sqrt{n}} = \dfrac{1}{n^{1/2}} \hookrightarrow p = \frac{1}{2} < 1$, diverges

$$\lim_{n\to\infty}\left(\frac{\frac{1}{\sqrt{n}+1}}{\frac{1}{\sqrt{n}}}\right) = \lim_{n\to\infty}\left(\frac{\cancel{1}}{n^{1/2}+1}\cdot\frac{n^{1/2}}{\cancel{1}}\right)$$

here I multiplied by this which is the same as multiplying by 1

$$= \lim_{n\to\infty}\left(\frac{n^{1/2}}{(n^{1/2}+1)}\right)\left(\frac{\frac{1}{n^{1/2}}}{\frac{1}{n^{1/2}}}\right) = \lim_{n\to\infty}\left(\frac{1}{1+\cancel{\frac{1}{n^{1/2}}}}\right)^{\to 0} = \boxed{1}$$

I did this because I wanted to get rid of my n term in the numerator & make it easier to evaluate the limit

Since lim is finite, & b_n diverges, a_n diverges

- **working with factorials:**
  - a factorial operator is the product of all positive numbers less than or equal to n:

$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \ldots$

or $n! = n \cdot (n-1)!$ $\qquad$ note: $0! = 1$

ex: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

note: factorials grow really fast

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- you sometimes see factorials in series

ex: $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^n (n+1)!}{n}}_{a_n}$

* in general, I like to use the ratio test for these types of series

$$\rightarrow p = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{(n+2)!}{(n+1)}}{\frac{(n+1)!}{n}} = \lim_{n\to\infty} \frac{(n+2)!}{(n+1)} \cdot \frac{n}{(n+1)!} \longrightarrow \text{simplify } \frac{(n+2)!}{(n+1)!}$$

$$\frac{(n+2)!}{(n+1)!}: \quad \underset{n=1}{\frac{3!}{2!}} = \frac{3 \cdot \cancel{2}}{\cancel{2}} \qquad \underset{n=2}{\frac{4!}{3!}} = \frac{4 \cdot \cancel{3} \cdot \cancel{2}}{\cancel{3} \cdot \cancel{2}} = 4 \qquad \underset{n=3}{\frac{5!}{4!}} = \frac{5 \cdot \cancel{4} \cdot \cancel{3} \cdot 2}{\cancel{4} \cdot \cancel{3} \cdot 2} = 5$$

$$\downarrow$$

$$\frac{(n+2)!}{(n+1)!} = n+2$$

$$= \lim_{n\to\infty} \frac{(n+2)(n)}{(n+1)} = \lim_{n\to\infty} \frac{n^2+2n}{n+1} \underset{\text{l'hopital}}{=} \lim_{n\to\infty} \frac{2n+2}{1} = \boxed{\infty, \text{ diverges}}$$

tricks for evaluating limits

- 1st: see if you can simplify it.

ex: $$\lim_{x\to 2} \frac{x^2+4x-12}{x^2-2x} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+6)}{x\cancel{(x-2)}} = \lim_{x\to 2} \frac{x+6}{x} = 4$$

- 2nd: see if you can manipulate it.

ex: $$\lim_{x\to\infty} \frac{3x^2-4}{5x-2x^2} = \frac{\infty}{\infty} \rightarrow \text{can use l'hopitals rule or put in different form}$$

$$\hookrightarrow \lim_{x\to\infty} \frac{\underbrace{x^2}_{\text{take out highest power \& factor}}\left(3-\frac{4}{x^2}\right)}{x^2\left(\frac{5}{x}-2\right)} = \lim_{x\to\infty} \frac{3-\frac{4}{x^2}}{\frac{5}{x}-2} \quad \text{now look at each term seperately} \longrightarrow \lim_{x\to\infty} \frac{3-\cancel{\frac{4}{x^2}}^{\,0}}{\cancel{\frac{5}{x}}^{\,0}-2} = \boxed{-\frac{3}{2}}$$

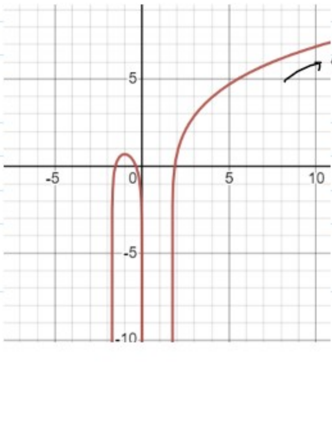
↪ recall: if numerator has higher power $\lim_{x\to\infty} = \infty$

if denominator has higher power $\lim_{x\to\infty} = 0$

- 3rd can you graph it/make a table to see what's happening at the limit?

$$\rightarrow \lim_{x\to\infty} \ln(x^3-3x)$$

in desmos:



goes to ∞

to check algebraically:

$$\lim_{x \to \infty} \ln(x^3 - 3x)$$

$$\lim_{x \to \infty} \ln\left(\cancel{x^3}^{\infty}\left(1 - \cancel{\frac{1}{3x^2}}^{0}\right)\right)$$

$$\ln(\infty) = \infty \checkmark$$