5.1 Sequences

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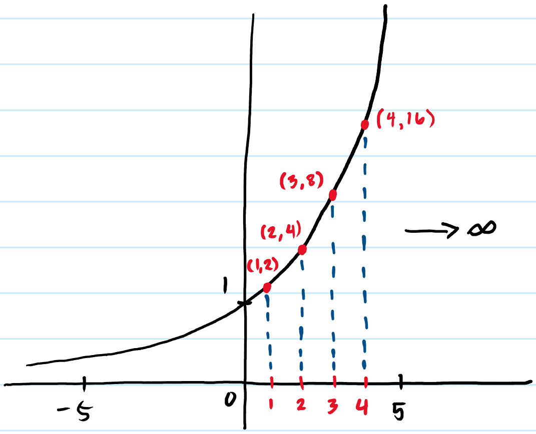
Objective for today:

1. What is a sequence? Definition & characteristics
2. Convergent vs divergent sequences

Introduction:

Consider the function $f(x) = 2^x$.



Consider limiting the domain to the natural numbers.

$$\left.\begin{aligned} n=1 &\to 2^1 \to 2 \\ n=2 &\to 2^2 \to 4 \\ n=3 &\to 2^3 \to 8 \\ n=4 &\to 2^4 \to 16 \\ &\vdots \\ \text{as } n \to \infty&, \ 2^n \to \infty \end{aligned}\right\} \text{infinite sequence}$$

We can write this sequence as

$$\{2, 4, 8, 16, \dots\} = \{2^n\}.$$

In general,

$a_n \to$ general term of a sequence

$\{a_n\} \to$ the sequence

$n \to$ index variable or the nth term.

Definition: A sequence or infinite sequence is an ordered list of numbers called terms.
A generic sequence $\{a_n\}$, where n is the

an ordered list of numbers called terms.
A generic sequence $\{a_n\}$, where n is the index variable.

Example:

infinite sequence

$$\{1 + 1/n\}_{n=1}^{\infty} = \{1 + 1/1,\ 1 + 1/2,\ 1 + 1/3, \ldots\}$$

sequence   index variable $n=1$ $n=2$ $n=3$

$$= \{2,\ 3/2,\ 4/3, \ldots\}$$

$a_n = 1 + 1/n \rightarrow$ general term
$\{a_n\} = \{1 + 1/n\} = \{2, 3/2, 4/3, \ldots\} \rightarrow$ sequence
$n = 1, 2, 3, \ldots \rightarrow$ index variable

Characteristics:

1. Arithmetic Sequence:
   - has a common difference between every pair of consecutive terms.

   ex. $\{3, 7, 11, 15, \ldots\}$

   $4 \quad 4 \quad 4 \quad \ldots$ common difference is 4

2. Geometric Sequence:
   - has a common ratio between every pair of consecutive terms.

   ex. $\{5,\ 5/2,\ 5/4,\ 5/8, \ldots\}$

   $\frac{5/2}{5} \quad \frac{5/4}{5/2} \quad \frac{5/8}{5/4}$

   $\downarrow \quad \downarrow \quad \downarrow$

   $1/2 \quad 1/2 \quad 1/2 \quad \ldots$ common ratio is $1/2$

3. Convergent vs Divergent sequences

   ex. $\{5/n\}$

$a_n = 5/n \rightarrow$ general term

$\{a_n\} = \{5/n\} = \{5/1, 5/2, 5/3, 5/4, \ldots\} \rightarrow$ the sequence

$$\lim_{n \to \infty} 5/n = 5 \lim_{n \to \infty} \frac{1}{n} = 5(0) = 0$$

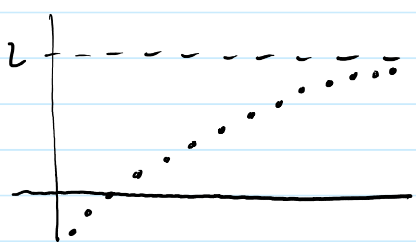
A <u>convergent</u> sequence is a sequence with terms that get closer and closer to a finite number L as the index variable goes to infinity.

The sequence $\{a_n\}$ is convergent if when $n \to \infty$, $a_n \to L$,

OR

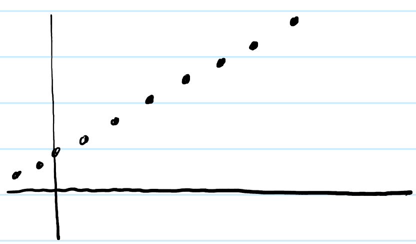
$$\lim_{n \to \infty} a_n = L.$$

ex.



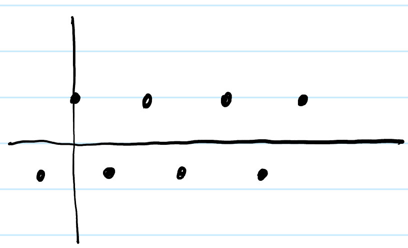
convergent

ex.



divergent

ex.



divergent

<u>Limit Laws</u>

Suppose $\{a_n\}$ and $\{b_n\}$ converge to A and B, c is a constant.

(a) $\lim_{n\to\infty} c = c$

(b) $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n = cA$

(c) $\lim_{n\to\infty} a_n \pm b_n = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$

(d) $\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = A \cdot B$

Other tools:

1. L'Hopital's Rule (Indeterminate forms $\frac{\infty}{\infty}$ or $\frac{0}{0}$)
2. Apply Natural log to both sides (Indeterminate form 1^{∞})

<u>Mini-Activity</u> (Think-Pair-Share)

Determine whether the sequences converge.
If convergent, find its limit.

(a) $\left\{\frac{(10)2^n}{n+1}\right\}_{n=1}^{\infty} = \left\{\underset{n=1}{\frac{(10)2^1}{1+1}}, \underset{n=2}{\frac{(10)2^2}{2+1}}, \underset{n=3}{\frac{(10)2^3}{3+1}}, \dots\right\}$

$= \{10, 40/3, 20, \dots\}$

$$\lim_{n\to\infty} \frac{(10)2^n}{n+1} \underset{\frac{\infty}{\infty}\text{ L'hopital's rule}}{=} \lim_{n\to\infty} \frac{d/dn\left[(10)2^n\right]}{d/dn(n+1)}$$

$$= \lim_{n\to\infty} \frac{(10)2^n \ln(2)}{1} \to \infty$$

$\therefore$ Divergent

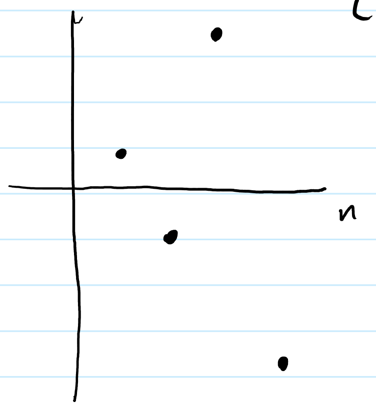
(b) $\{(-3)^{n-1}\}_{n=1}^{\infty} = \{\underset{n=1}{(-3)^{1-1}}, \underset{n=2}{(-3)^{2-1}}, \underset{n=3}{(-3)^{3-1}}, \dots\}$

$= \{(-3)^0, (-3)^1, (-3)^2, (-3)^3, \dots\}$

$$(b)\ \{(-3)^{n-1}\}_{n=1}^{\infty} = \{(-3)^{1-1}, (-3)^{2-1}, (-3)^{3-1}, \ldots\}$$

$$= \{(-3)^0, (-3)^1, (-3)^2, (-3)^3, \ldots\}$$

$$= \{1, -3, 9, -27, \ldots\}$$



The sequence is alternating and growing larger.

$\therefore$ Divergent

$$(c)\ \left\{\left(\frac{e}{2}\right)^n\right\}_{n=1}^{\infty} = \left\{ \underset{n=1}{\left(\frac{e}{2}\right)^1}, \underset{n=2}{\left(\frac{e}{2}\right)^2}, \underset{n=3}{\left(\frac{e}{2}\right)^3}, \ldots \right\}$$

$$= \{e/2, e^2/4, e^3/8, \ldots\}$$

$$\lim_{n\to\infty} (e/2)^n = L \longrightarrow \lim_{n\to\infty} n \ln(e/2) = \ln(L)$$

$$\lim_{n\to\infty} n(\ln(e) - \ln(2)) = \ln(L)$$

$$\lim_{n\to\infty} n(1 - \ln(2)) = \ln(L)$$

$$\underbrace{\lim_{n\to\infty} n - \lim_{n\to\infty} n \ln(2)}_{\text{undefined}} = \ln(L)$$

$\therefore$ Divergent

$$(d)\ \left\{\frac{n-4}{3n+1}\right\}_{n=1}^{\infty} = \left\{ \underset{n=1}{\frac{1-4}{3(1)+1}}, \underset{n=2}{\frac{2-4}{3(2)+1}}, \underset{n=3}{\frac{3-4}{3(3)+1}}, \underset{n=4}{\frac{4-4}{3(4)+1}}, \ldots \right\}$$

$$= \left\{ -\tfrac{3}{4}, -\tfrac{2}{7}, -\tfrac{1}{10}, \tfrac{0}{13}, \dots \right\}$$

$$\lim_{n\to\infty} \frac{n-4}{3n+1} \underset{\text{L'Hopital's rule}}{=} \lim_{n\to\infty} \frac{d/dn[n-4]}{d/dn[3n+1]} = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3}$$

$\therefore$ Convergent

(e) $\left\{ \sqrt{n^2+5n} - n \right\}_{n=1}^{\infty} = \left\{ \underset{n=1}{\sqrt{1^2+5(1)} - 1}, \underset{n=2}{\sqrt{2^2+5(2)} - 2}, \underset{n=3}{\sqrt{3^2+5(3)} - 3}, \dots \right\}$

$$= \left\{ \sqrt{6} - 1, \sqrt{14} - 2, \sqrt{42} - 3, \dots \right\}$$

$$\lim_{n\to\infty} \sqrt{n^2+5n} - n = \lim_{n\to\infty} \frac{(\sqrt{n^2+5n} + n)(\sqrt{n^2+5n} - n)}{(\sqrt{n^2+5n} + n)}$$

$$= \lim_{n\to\infty} \frac{(n^2+5n) - \cancel{n\sqrt{n^2+5n}} + \cancel{n\sqrt{n^2+5n}} - n^2}{\sqrt{n^2+5n} + n}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2} + 5n - \cancel{n^2}}{\sqrt{n^2+5n} + n}$$

Apply L'hopital's rule

$$= \lim_{n\to\infty} \frac{5n}{\sqrt{n^2+5n} + n}$$

$$= \lim_{n\to\infty} \frac{5}{\frac{2n+5}{2\sqrt{n^2+5n}} + 1}$$

$$\frac{2n+5}{\sqrt{4(n^2+5n)}} = \sqrt{\frac{(2n+5)^2}{4(n^2+5n)}}$$

$$= \frac{1}{2}\sqrt{\frac{(2n+5)^2}{n(n+5)}}$$

$$= \lim_{n\to\infty} \frac{5}{1+1}$$

$$= 5/2$$

$\therefore$ Convergent

$$\lim_{n\to\infty} \frac{1}{2}\sqrt{\frac{(2n+5)^2}{n(n+5)}} = L$$

$$\frac{1}{2} \lim_{n\to\infty} \sqrt{\frac{(2n+5)^2}{n(n+5)}} = L$$

L'hopital's rule

$$\frac{1}{4} \lim_{n\to\infty} \frac{(2n+5)^2}{n(n+5)} = L^2$$

$$\frac{1}{4} \lim_{n\to\infty} \frac{4(2\cancel{n+5})}{(2\cancel{n+5})} = L^2$$

$$\frac{1}{2} \lim_{n\to\infty} \sqrt{4} = L$$

$$\frac{1}{2} \lim_{n \to \infty} \sqrt{4} = L$$

$$\frac{1}{2} \, 2 = L = 1$$

(f) $\left\{ \frac{n!}{(n+1)!} \right\}_{n=1}^{\infty} = \left\{ \overset{n=1}{\frac{1!}{2!}}, \overset{n=2}{\frac{2!}{3!}}, \overset{n=3}{\frac{3!}{4!}}, \dots \right\}$

$= \left\{ 1/2, \; 2/6, \; 6/24, \; \dots \right\}$

$$\lim_{n \to \infty} \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{\cancel{n!}}{\cancel{n!}(n+1)} = \lim_{n \to \infty} \frac{1}{n+1}$$

consider $(n+1)! = n!(n+1)$ $\qquad = 0$

$\therefore$ Convergent