5.1 Sequences & 5.2 Series

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Quijano, AJ

Objectives:

1. Find the formula for the general term of a sequence.
2. Monotone Convergence theorem
3. the Squeeze theorem
4. Introduction to Series

Previously on Series

1. the sequence $\{a_n\}$ is convergent if when $n \to \infty$, $a_n \to L$.

   OR $\lim_{n \to \infty} a_n = L$.

2. Two characteristics
   - Arithmetic sequence ⇒ common difference between every pair of consecutive terms.
   - Geometric sequence → common ratio between every pair of consecutive terms.

How to find the nth term of a arithmetic sequence?

Formula for the nth term:

$$a_n = a_1 + (n-1)d$$

nth term, first term, nth index, common difference

Justification example:

- Consider the sequence

$$3, 7, 11, 15, 19, \ldots \quad d = 4$$

$$a_1, a_2, a_3, a_4, a_5$$

$$a_{\boxed{1}} = \boxed{3}$$

$$a_{\boxed{2}} = \boxed{3} + \boxed{1}\boxed{4} = 7$$

$$a_{\boxed{3}} = 3 + 4 + 4 = \boxed{3} + \boxed{2}\boxed{(4)} = 11$$

$$a_{\boxed{4}} = 3 + 4 + 4 + 4 = \boxed{3} + \boxed{3}\boxed{(4)} = 15$$

$$a_{\boxed{5}} = 3 + 4 + 4 + 4 + 4 = \boxed{3} + \boxed{4}\boxed{4)} = 19$$

$$\vdots$$

$$a_n = 3 + (n-1)4$$

Recurrence relation

$a_n = a_{n-1} + 4$

where $a_0 = -1$.

nth term formula $\rightarrow$ $a_n = 3 + (n-1)4 = 4n-1$

How to find the nth term of a geometric sequence?

Formula for the nth term:

$$\underbrace{a_n}_{\text{nth term}} = \underbrace{a_1}_{\text{first term}} \underbrace{r}_{\text{common ratio}}^{n-1}$$

In general, geometric sequences is any sequence of the form $a_n = c r^{n-1}$ (start $n=1$) or $a_{n+1} = c r^n$ (start $n=0$)

Justifiction for the formula:
- Consider the sequence

$$\underbrace{2}_{a_1}, \underbrace{2/3}_{a_2}, \underbrace{2/9}_{a_3}, \underbrace{2/27}_{a_4}, \underbrace{2/81}_{a_5}, \ldots \quad r = 1/3$$

$a_1 = 2$
$a_2 = (1/3)2$
$a_3 = (1/3)(1/3)2 = (1/3)^2 2$
$\vdots$

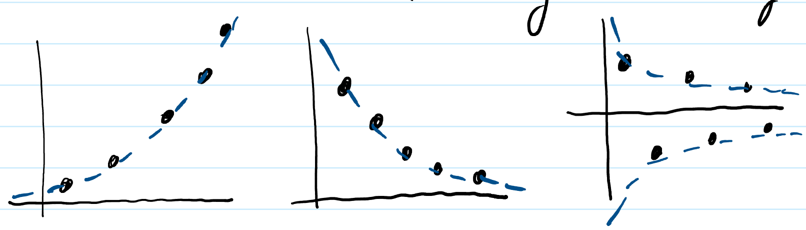
$a_n = 2\,(1/3)^{n-1}$, starting $n=1$

$a_{n+1} = 2\,(1/3)^n$, starting $n=0$

Question! What if a sequence is neither arithmetic nor geometric?
Look for patterns by algebraic manipulations

Monotone Sequences

A sequence is said to be monotone if it is either increasing or decreasing.



Monotone vergence theorem

- If a s ce is increasing & bounded above, then verges.
- If a se is decreasing & bounded below, th onverges.

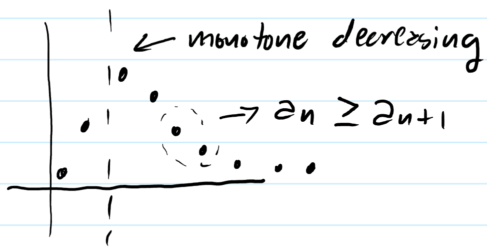
ex.

convergent monotonic sequence

monotonic sequence + increasing

- Additional te: If at some po l in sequence te terms are only incr ng/decre ing the sequence is monotone

ex.



Squeeze theorem

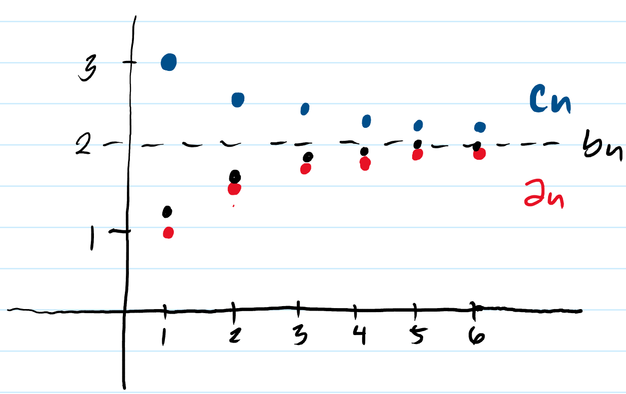
Consider sequences $\{a_n\}$, $\{b_n\}$, & $\{c_n\}$.
Suppose $\exists N$ such that $a_n \leq b_n \leq c_n \ \forall \ n \geq N$.

If $\exists L \in \mathbb{R}$ such that

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$$

then $\{b_n\}$ converges and $\lim\limits_{n\to\infty} b_n = L$.

ex.
Find $\displaystyle\lim_{n\to\infty} \frac{2n - \sin(n)}{n}$



We know that : $-1 \leq \sin(n) \leq 1$
multiply by -1 : $1 \geq -\sin(n) \geq -1$
$-1 \leq -\sin(n) \leq 1$

built up to expression

$$\frac{2n-1}{n} \le \frac{2n-\sin(n)}{n} \le \frac{2n+1}{n}$$

$$a_n \le b_n \le c_n$$

$$\lim_{n\to\infty} \frac{2n-1}{n} \le \lim_{n\to\infty} \frac{2n-\sin(n)}{n} \le \lim_{n\to\infty} \frac{2n+1}{n}$$

By the Squeeze theorem,

$$2 \le \lim_{n\to\infty} \frac{2n-\sin(n)}{n} \le 2$$

$$\lim_{n\to\infty} \frac{2n-1}{n} = 2 = \lim_{n\to\infty} \frac{2n+1}{n}$$

$\therefore$ the sequence converges to 2.

Introduction to Series

A series is the sum of a sequence denoted as $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$

ex.

Infinite sequence : $\{2+5^n\}_{n=1}^{\infty} = \{7, 27, 127, \ldots\}$

Infinite series : $1 + 27 + 127 + \cdots = \sum_{n=1}^{\infty} 2+5^n$

→ upper limit

→ lower limit

summation notation called sigma

Geometric series :

$\{2(1/3)^{n-1}\}_{n=1}^{\infty} = \{2, 2/3, 2/9, 2/27, 2/81, \ldots\}$

↳ sequence

How to write this in series from?

form 1 : $\sum_{n=1}^{\infty} 2(1/3)^{n-1}$

form 2 : $\sum_{n=0}^{\infty} 2(1/3)^{n}$

In general,

form 1 : $\sum^{\infty} cr^{n-1}$

In general,

form 1: $\sum_{n=1}^{\infty} cr^{n-1}$

form 2: $\sum_{n=0}^{\infty} cr^{n}$

Kth Partial Sum:

For each positive integer k, the sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \ldots + a_k$$

for a sequence $\{a_n\}$.

1. If the sequence of partial sums converges to a real number S, then the infinite series converges.
2. If the sequence of partial sums diverges, then the series diverges.

Geometric series: $\sum_{n=1}^{\infty} 2\left(\frac{1}{3}\right)^{n-1}$

Find the kth partial sum.

$$S_1 = \sum_{n=1}^{1} 2\left(\tfrac{1}{3}\right)^{n-1} = 2$$

$$S_2 = \sum_{n=1}^{2} 2\left(\tfrac{1}{3}\right)^{n-1} = 2 + \tfrac{2}{3}$$

$$S_3 = \sum_{n=1}^{3} 2\left(\tfrac{1}{3}\right)^{n-1} = 2 + \tfrac{2}{3} + \tfrac{2}{9}$$

$$S_4 = \sum_{n=1}^{4} 2\left(\tfrac{1}{3}\right)^{n-1} = 2 + \tfrac{2}{3} + \tfrac{2}{9} + \tfrac{2}{27}$$

$$\vdots$$

$$S_k = \sum_{n=1}^{k} 2\left(\tfrac{1}{3}\right)^{n-1} = 2\left(1 + \tfrac{1}{3} + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{3}\right)^3 + \ldots + \left(\tfrac{1}{3}\right)^{k-1}\right)$$

$$\left(1-\tfrac{1}{3}\right)S_k = 2\left(1-\tfrac{1}{3}\right)\left(1 + \tfrac{1}{3} + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{3}\right)^3 + \ldots + \left(\tfrac{1}{3}\right)^{k-1}\right)$$

$$= 2\left[\left(1 + \tfrac{1}{3} + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{3}\right)^3 + \ldots + \left(\tfrac{1}{3}\right)^{k-1}\right) - \left(\tfrac{1}{3} + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{3}\right)^3 + \ldots + \left(\tfrac{1}{3}\right)^{k}\right)\right]$$

$$= 2\left(1 - \left(\tfrac{1}{3}\right)^k\right)$$

$$S_k = \frac{2\left(1 - \left(\frac{1}{3}\right)^k\right)}{1 - \frac{1}{3}}$$

In general, the kth partial sum of a geometric series of the form $\sum_{n=1}^{\infty} ar^{n-1}$ is

$$S_k = \frac{a(1-r^k)}{1-r} \quad \text{for } r \neq 1$$

Mini-Activities

For the following sequences,

a. Find the explicit form for a_n. Assume n starts at 1.
b. Determine if the sequence is convergent or divergent.
c. Write the inf. series using sigma notation and its kth partial sum

(1.) $\{8, 13, 18, 23, 28, \dots\}$

$$\underbrace{8,13}_{5}\;\underbrace{13,18}_{5}\;\underbrace{18,23}_{5}\;\underbrace{23,28}_{5}$$

common difference is $d=5$

$a_1 = 8$
$a_2 = 8+5 = 8+(1)5 = 13$
$a_3 = 8+5+5 = 8+(2)5 = 18$
$a_4 = 8+5+5+5 = 8+(3)5 = 23$
$\vdots$ $\qquad n-1$
$a_n = 8+(n-1)5$

The pattern is $\{a_n\} = \{8+(n-1)5\}$

$\lim_{k\to\infty} 8+(k-1)5 = \text{DNE} \to$ divergent seq.

Series, $\sum_{n=1}^{\infty} 8+(n-1)5 = 8+13+18+23+\dots$
$\hookrightarrow$ divergent series

(2.) $\{2, -6, 18, -54, \dots\}$

$-6/2 = -3 \qquad 18/-6 = -3 \qquad -54/18 = -3$

common ratio is $r = -3$

this is a geometric series of the form $a_n = 2(-3)^{n-1}$

Series: $\sum_{n=1}^{\infty} 2(-3)^{n-1} = 2-6+18-54+\dots$
$\to$ divergent series

kth partial sum:

$$S_1 = \sum_{n=1}^{1} 2(-3)^{n-1} = 2 = \frac{2(1-(-3)^1)}{1-(-3)}$$

$$S_1 = \sum_{n=1}^{1} 2(-3)^{n-1} = 2 = \frac{2(1-(-3)^1)}{1-(-3)}$$

$$S_2 = \sum_{n=1}^{2} 2(-3)^{n-1} = 2-6 = \frac{2(1-(-3)^2)}{1-(-3)}$$

$$\vdots$$

$$S_k = \sum_{n=1}^{k} 2(-3)^{n-1} = \frac{2(1-(-3)^k)}{1-(-3)} \rightarrow \text{still divergent}$$

(3.) $\{3, -3, 3, -3, 3, \dots\}$

common ratio is $r = -1$

This is a geometric series,

so,

$$a_n = 3(-1)^{n-1}$$

and the sequence diverges.

The sigma notation for the series is

$$\sum_{n=1}^{\infty} 3(-1)^{n-1}$$

which is a divergent series.

Kth partial sum is $S_k = \frac{3(1-(-1)^k)}{1-(-1)} \rightarrow$ still divergent

(4.) $\{2, 3/2, 5/4, 9/8, 17/16, 33/32\}$

(5.) $\{2, 5/2, 13/4, 33/8, 81/16, 193/2\}$

(6.) $a_1 = 6$, $a_n = \frac{a_{n-1}}{3}$

(7.) $a_1 = 6$, $a_n = -\frac{a_{n-1}}{3}$

(8.) $a_1 = 6$, $a_{n-1} + 3$

(9.) $a_1 = 2$, $a_n = -2a_{n-1}$

(10.) $a_1 = 2$, $a_n = a_{n-1} + 3$