5.2 Series & 5.3 Divergence Test

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Quijano, AJ

Objectives:

1. Geometric series (conv. & Dive. cases)
2. Telescoping series
3. Divergence test introduction

Previously...

1. General form of the geometric series

Sequence: $A_n = a_1 r^{n-1}$

series: $\sum_{n=1}^{\infty} a_n = a_1 r^{n-1}$

where r is the common ratio.

2. The kth partial sum of the geometric series is

$$S_k = \frac{a(1-r^k)}{1-r} \text{ for } r \neq 1.$$

Convergence & Divergence of the geometric series.

Consider the general term for the geometric sequence given as

$$\{r^n\}.$$

Here are some propositions.

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1. If $-1 < r < 1$, then $\lim\limits_{n\to\infty} r^n = 0$.

   So, $\{r^n\}$ converges if $0 < r < 1$.

2. If $|r| = 1$, then $\lim\limits_{n\to\infty} r^n = 1$.

   So, $\{r^n\}$ has a limit of 1.

3. If $|r| > 1$, then $\lim\limits_{n\to\infty} r^n = \text{DNE or } \infty$

   So, $\{r^n\}$ diverges if $r > 1$.

Back to the kth partial sum.

$$S_k = \frac{a(1-r^k)}{1-r} \quad \text{for } r \neq 1.$$

If $-1 < r < 1$ or $|r| < 1$, then

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{a(1-r^k)}{1-r}$$

$$= \frac{a}{1-r} \lim_{k\to\infty} 1 - r^k$$

$$= \frac{a}{1-r}\Big(\lim_{k\to\infty} 1 - \underbrace{\lim_{k\to\infty} r^k}_{0}\Big)$$

$$= \frac{a}{1-r}, \text{ the series converges}$$

Thus,

$$S = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ if } |r| > 1$$

If $|r| \geq 1$, the series $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

ex. $\sum_{n=1}^{\infty} 2\left(-\frac{1}{5}\right)^{n-1}$, $r = -1/5$

Since $|r| < 1$, then it converges

the kth partial sum is

$$S_k = \frac{2(1-(-1/5)^k)}{1-(-1/5)} = \frac{2(1-(-1/5)^k)}{1+1/5}$$

the infinite sum is

$$S = \frac{2}{1+1/5} = \frac{2}{6/5} = \frac{10}{6} = \frac{5}{3}.$$

Telescoping Series

1. Consider this sequence from last time

$$\{3, -3, 3, -3, 3, \ldots\} = \{3(-1)^{n-1}\}$$

common ratio

Notice the alternating signs & algebraic cancelations.
Since $|r| = 1$, then the sequence diveges.
So, $\sum_{n=1}^{\infty} 3(-1)^{n-1}$ diverges.

kth partial sum is

$$S_k = \sum_{n=1}^{k} 3(-1)^{n-1}$$

$$\sum_{n=1} 3(-1)$$

$$= \frac{3(1-(-1)^k)}{1+1} = \frac{3(1-(-1)^k)}{2}$$

but the series diverges,

because $\lim_{k\to\infty} \frac{3}{2}(1-(-1)^k) \to$ DNE.

2. $$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots$$

$n=1 \qquad n=2 \qquad n=3$

Determining the kth partial sum:

$$S_1 = 1 - \tfrac{1}{2}$$
$$S_2 = 1 - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}} - \tfrac{1}{3} = 1 - \tfrac{1}{3}$$
$$S_3 = 1 - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{3}} - \tfrac{1}{4} = 1 - \tfrac{1}{4}$$
$$\vdots$$
$$S_k = 1 - \frac{1}{k+1}.$$

Infinite Sum:

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} 1 - \frac{1}{k+1} = 1 - 0 = 1$$

Thus, the telescoping series is convergent.

Note that if the series of partial sums diverges, then the series diverges.

Theorem:

If the series of partial sums converges - that is $\lim_{k\to\infty} \sum_{n=1}^{k} a_n$ converges, then the $\lim_{n \to \infty} a_n = 0$.

Note that the converse does not hold.

Divergence test

If $\lim_{n \to \infty} a_n \neq 0$ or the $\lim_{n \to \infty} a_n \to$ DNE, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Note! If $\lim_{n \to \infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ may or may not converge $\to$ inconclusive.

- Determine if the series diverges or converges. using the divergence test.

ex. $\sum_{n=1}^{\infty} \underbrace{\frac{4n^2 - n^3}{10 + 2n^3}}_{a_n} \to \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{4n^2 - n^3}{10 + 2n^3}$

L'hopital's rule

$$= \lim_{n \to \infty} \frac{8n - 3n^2}{6n^2}$$

$$= \lim_{n \to \infty} \frac{8 - 6n}{12n}$$

$$= \lim_{n \to \infty} \frac{-6}{12}$$

$$= \frac{-1}{2}$$

So, $\lim_{n \to \infty} a_n \neq 0$, then the series diverges.

ex. $\sum_{n=1}^{\infty} \frac{1}{n} \rightarrow \lim_{n\to\infty} \frac{1}{n} = 0$

thus, the divergence test is inco<u>nclusi</u>ve.

Harmonic series, which is actualy divergent but why?
The <u>integral test</u> will shed some light on this
but more on this later on.

<u>Mini-Activity</u>

(1.) Determine whether the series below converges or diverges. If it converges find its sum.

(a.) $\sum_{n=1}^{\infty} \left(-\frac{2}{5}\right)^{n-1}$

$\lim_{n\to\infty} \left(-\frac{2}{5}\right)^{n-1}$ which is of the form $\lim_{n\to\infty} ar^{n-1}$ where $r = -2/5$

Since $|r| < 1$, then $\lim_{n\to\infty} \left(\frac{-2}{5}\right)^{n-1} = 0$.

By the divergence test tee, the result is inconclusive.
However, this is a geometric series with $|r| < 1$.
So, we know that
$$\sum_{n=1}^{\infty} \left(-\frac{2}{5}\right)^{n-1} \text{ converges since } |r| < 1.$$

and $$S = \sum_{n=1}^{\infty} \left(-\frac{2}{5}\right)^{n-1} = \frac{1}{1-(-2/5)} = \frac{5}{7}$$

(b) $\sum_{n=1}^{\infty} \left[e^{1/n} - e^{1/(n+1)}\right]$

$$\lim_{n\to\infty} e^{1/n} - e^{1/(n+1)} = \cancel{\lim_{n\to\infty} e^{1/n}}^{1} - \cancel{\lim_{n\to\infty} e^{1/(n+1)}}^{1} = 0$$

The divergence test is inconclusive.
kth partial sum:

$$S_1 = \sum_{n=1}^{1} e^{1/n} - e^{1/n+1} = e - e^{1/2}$$

$$S_2 = \sum_{n=1}^{2} e^{1/n} - e^{1/n+1} = e - \cancel{e^{1/2}} + \cancel{e^{1/2}} - e^{1/3} = e - e^{1/3}$$

$$S_3 = \sum_{n=1}^{3} e^{1/n} - e^{1/n+1} = e - \cancel{e^{1/2}} + \cancel{e^{1/2}} - \cancel{e^{1/3}} + \cancel{e^{1/3}} - e^{1/4} = e - e^{1/4}$$

$$\vdots$$

$$S_k = \sum_{n=1}^{k} e^{1/n} - e^{1/n+1} = e - e^{1/k+1}$$

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} e - e^{\frac{1}{k+1}} = \lim_{k\to\infty} e - \cancel{\lim_{k\to\infty} e^{1/k+1}}^{1} = e - 1$$

thus, the series converges to $e-1$.

(2.) Use the Divergence test to determine if the following series converges or diverges, or inconclusive

(a) $\sum_{n=1}^{\infty} \frac{n}{3n-1}$

(a) $\displaystyle\sum_{n=1} \underbrace{\frac{n}{3n-1}}_{a_n}$ L'hospital

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{3n-1} = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3} \neq 0$$

By the divergence test, $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3n-1}$ diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \underbrace{e^{1/n^2}}_{a_n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{1/n^2} = 1 \neq 0$$

By the divergenc test, $\displaystyle\sum_{n=1}^{\infty} e^{1/n^2}$ diverges.

(c) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{2^n}{3^{n/2}}}_{a_n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n}{3^{n/2}}$$

L'hospital $\hookrightarrow = \displaystyle\lim_{n\to\infty} 3^{n/2}\left(\frac{2}{3}\right)^n \to$ DNE

By the divergence test, $\displaystyle\sum_{1}^{\infty} 2^n$ diverges.

By the divergence test, $\sum_{n=1}^{\infty} \frac{2^n}{3^{n/2}}$ diverges.