3.7 Improper Integrals & Sequence and Series Review

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Objectives:

1. Review on Sequence & series.
2. What makes an integral "improper"?

Sequences & Series example problem

1. $a_n = \pi + \sin\left(\frac{2\pi}{n^2}\right)$

Does the sequence converge?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \pi + \sin\left(\frac{2\pi}{n^2}\right)$$

goes to 0

$$= \pi + \lim_{n\to\infty} \sin\left(\frac{2\pi}{n^2}\right)$$

$$= \pi + \sin(0)$$

$$= \pi$$

The sequence $a_n = \pi + \sin\left(\frac{2\pi}{n^2}\right)$ converge.

2. $a_n = \frac{\sin(2x) + 2}{2x}$

Does the sequence converge?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin(2x)+2}{2x}$$

$\downarrow$

apply the squeeze theorem

Let $-1 \le \sin(2x) \le 1$.

$$\frac{-1+2}{2x} \le \frac{\sin(2x)+2}{2x} \le \frac{1+2}{2x}$$

$$\cancel{\lim_{x\to\infty} \frac{1}{2x}}^{\,0} \le \lim_{x\to\infty} \frac{\sin(2x)+2}{2x} \le \cancel{\lim_{x\to\infty} \frac{3}{2x}}^{\,0}$$

$$0 \le \lim_{x\to\infty} \frac{\sin(2x)+2}{2x} \le 0$$

By the squeeze theorem, the sequence a_n converges to 0.

3. $a_n = 5 + e^{-3n}$

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Does the sequence converge?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 5 + e^{-3n}$$

$$= \lim_{n\to\infty} 5 + \frac{1}{e^{3n}}$$

$$= 5 + \underbrace{\lim_{n\to\infty} \frac{1}{e^{3n}}}_{0}$$

$$= 5$$

The sequence converges to 5.

4. Monotonically inc. & decreasing seq.

a. $a_n = \frac{n-3}{n+3}$, Is this sequence increasing decreasing or not monotonic?

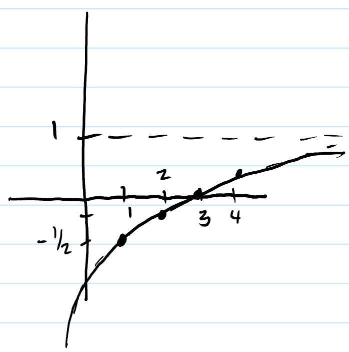
$a_1 = \frac{1-3}{1+3} = \frac{-2}{4} = -\frac{1}{2}$

$a_2 = \frac{2-3}{2+3} = \frac{-1}{5}$

$a_3 = \frac{3-3}{3+3} = \frac{0}{6} = 0$

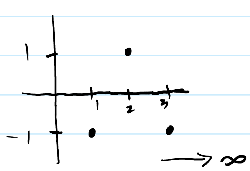
$a_4 = \frac{4-3}{4+3} = \frac{1}{6}$

⋮



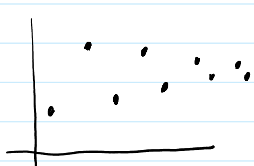
The seq. is increasing also converging.

b. $a_n = (-1)^n$



this seq. is not monotonic.

c. $a_n = 2 + \frac{(-1)^n}{n}$



↳ not monotonic but it converges.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{8n}{9-2n}$

Does the series converge?

Apply the divergence test

$a_n = \frac{8n}{9-2n} \rightarrow$ sequence term

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{8n}{9-2n}$$

$$= -\frac{8}{2} = -4 \neq 0$$

Thus, the series diverges ... but the sequence converges to -4.

Where does the series diverge to? $-\infty$

6. Summary:

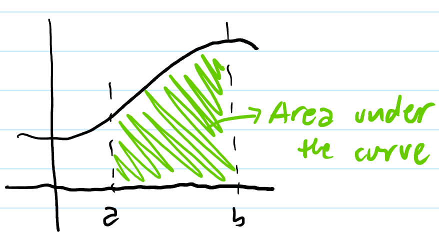
1. The relationship between sequences and series (notations, convergence)
2. kth partial sum
3. Divergence test
4. geometric series (divergent & convergent cases)
5. Telescoping series (check if kth partial sum converges to check convergence of the sequence)

<u>Improper Integrals</u>

Definite integral: $\int_a^b f(x)\,dx$

Indefinite integral: $\int f(x)\,dx$ $\downarrow$ aka anti-derivative

- proper: $f(x)$ is continuous and integrable at interval $[a,b]$.
  $\downarrow$
  $$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$
- improper
  - type I: The bounds are infinite.
  - type II: $f(x)$ is discontinuous at interval $[a,b]$



I. Improper Integral type I

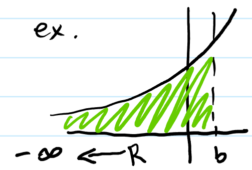
- The bounds are infinite.

1. if $f(x)$ is continuous on $[a, \infty)$,

$$\int_a^{\infty} f(x)\,dx = \lim_{R \to \infty} \int_a^R f(x)\,dx$$

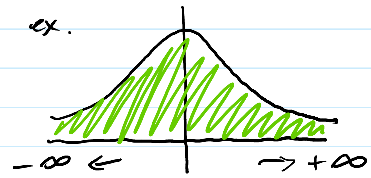
2. if $f(x)$ is continuous on $(-\infty, b]$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{R \to -\infty} \int_R^b f(x)\,dx$$



3. If the limits above are convergent, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx.$$



II. Improper Integrals type II.

- $f(x)$ is discontinuous on $[a, b]$

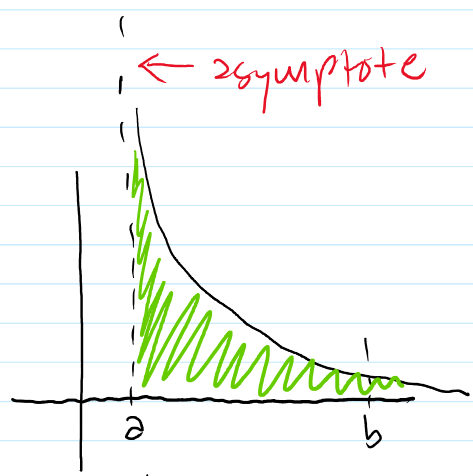
→ discontinuous at a

1. If $f(x)$ is continuous on $(a, b]$,

$$\int_a^b f(x)\,dx = \lim_{R \to a^+} \int_R^b f(x)\,dx$$

limit aproaching a from the right.

2. if $f(x)$ is continuous on $[a, b)$, ← discontinuous at b



$$\int_a^b f(x)\,dx = \lim_{R \to b^-} \int_a^R f(x)\,dx$$

limit approaching b from the left.

3. If $f(x)$ is discontinuous at $x = c$ where $c \in [a, b]$ and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Ex. $\int^{\infty}$

Ex. $\int_1^\infty \frac{1}{x}\,dx$, this is type I improper integral.

harmonic function

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx$$

$$= \lim_{R\to\infty} \ln(|x|)\Big|_1^R$$

$$= \lim_{R\to\infty} \ln(|R|) - \lim_{R\to\infty} \ln(|1|)$$

$$= \lim_{R\to\infty} \ln(|R|) - 1 = \text{DNE}$$

$\to +\infty$. thus, the integral diverges

__Mini-Activity.__

Determine the type of the
the following improper integrals Do not solve yet.

1. $\int_0^1 \ln(x)\,dx$

2. $\int_0^4 \frac{dx}{\sqrt{4-x}}$

3. $\int_1^2 \frac{dx}{(x-1)^{1/3}}$

4. $\int_1^\infty \frac{1}{x^2}\,dx$

5. $\int_{-\infty}^1 \frac{1}{1-x^2}\,dx$