3.7 Improper Integrals Cont. & 5.3 Integral Test

Friday, September 9, 2022

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Objectives

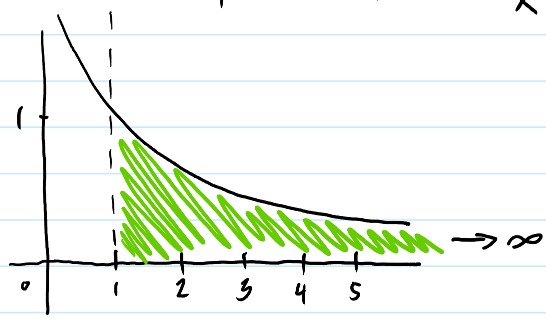
1. Identifying the type of improper integral.
2. Solving an improper integral examples
3. Integral test

Previously...

- Type I improper integral
  - the bounds are infinite
- Type II improper integral
  - $f(x)$ is discontinuous on $[a,b]$

<u>Solving an improper integral of Type I</u> examples

1. Consider the function $f(x) = \frac{1}{x}$



$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{R\to\infty} \int_1^{R} \frac{1}{x}\,dx$$

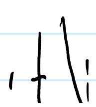
$$= \lim_{R\to\infty} \ln(|x|)\Big|_1^{R}$$

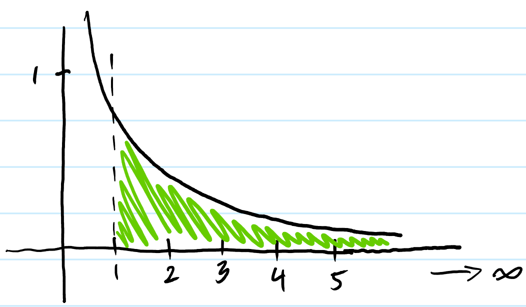
$$= \lim_{R\to\infty} \ln(|R|) - \cancel{\lim_{R\to\infty} \ln(|1|)}$$

this limit diverges

$$= DNE$$

2. $f(x) = \frac{1}{x^2}$





$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{R\to\infty}\int_1^R \frac{1}{x^2}\,dx$$

→ power rule

$$= \lim_{R\to\infty}\int_1^R x^{-2}\,dx$$

$$= \lim_{R\to\infty} \left.\frac{x^{-1}}{-1}\right|_1^R$$

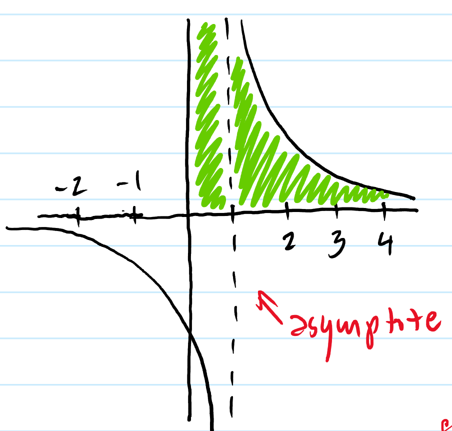
$$= \lim_{R\to\infty} \left.-\frac{1}{x}\right|_1^R$$

$$= \overset{0}{\cancel{\lim_{R\to\infty} -\frac{1}{R}}} - \overset{+1}{\cancel{\lim_{R\to\infty} -\frac{1}{1}}}$$

$$= 1$$

Solving an improper integral of type II examples

1. $f(x) = \dfrac{1}{x-1}$



← f(x) is discontinuous at x=1

$$\int_0^4 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^4 \frac{1}{x-1}\,dx$$

$$= \lim_{R\to 1^-}\int_0^R \frac{1}{x-1}\,dx + \lim_{R\to 1^+}\int_R^4 \frac{1}{x-1}\,dx$$

$$= \lim_{R\to 1^-} \ln(x-1)\Big|_0^R + \lim_{R\to 1^+} \ln(x-1)\Big|_R^4$$

$$= \overset{DNE}{\cancel{\lim_{R\to 1^-} \ln(|R-1|)}} - \overset{0}{\cancel{\lim_{R\to 1^-} \ln(|0-1|)}} + \overset{DNE}{\cancel{\lim_{R\to 1^+} \ln(|R-1|)}} - \overset{\ln(3)}{\cancel{\lim_{R\to 1^+} \ln(|4-1|)}}$$

$$= DNE$$

2. $f(x) = \dfrac{1}{\sqrt{4-x}}$

$$\int_0^4 \frac{1}{\sqrt{4-x}}\,dx = \lim_{R\to 4^-} \int_0^R \frac{1}{\sqrt{4-x}}\,dx$$

power rule

$$= \lim_{R\to 4^-} \int_0^R (4-x)^{-1/2}\,dx$$

$$= \lim_{R\to 4^-} -\frac{(4-x)^{1/2}}{1/2}\Big|_0^R$$

$$= \lim_{R\to 4^-} -2\sqrt{4-x}\,\Big|_0^R$$

$$= \overset{0}{\cancel{\lim_{R\to 4^-} -2\sqrt{4-R}}} - \lim_{R\to 4^-} -2\sqrt{4-0}$$

$$= 2(2)$$
$$= 4$$

<u>Integral Test</u>

- Consider the harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

1. the divergence test is conclusive on the harmonic series.
2. We know that the series diverges.
3. We showed that

$$\int_1^{\infty} \frac{1}{x}\,dx \to \text{DNE} \to \text{diverges.}$$

- Showing the kth partial sum of the series diverges.

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{3}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \longrightarrow \text{by comparison}$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4}$$

$$\vdots$$

$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$\underbrace{1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}}_{1 + 3(1/2)}$$

the pattern

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{3}$$

$$S_4 > 1 + 2\left(\frac{1}{2}\right)$$

$$S_8 > 1 + 3\left(\frac{1}{2}\right)$$

in general

$$S_{2^k} > 1 + k\left(\frac{1}{2}\right)$$

So, $\lim_{k \to \infty} 1 + k\left(\frac{1}{2}\right) \to \text{DNE} \to +\infty.$

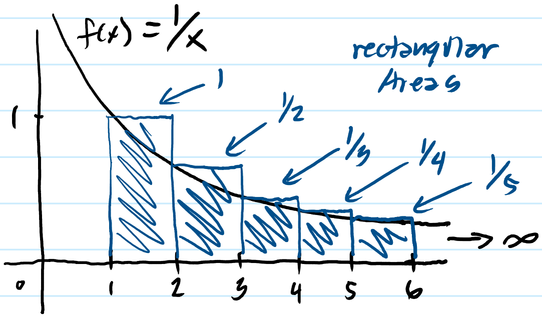
So, $\lim_{k\to\infty} 1 + k\left(\frac{1}{2}\right) \to DNE \to +\infty$.

the kth partial sum is unbounded.
Thus, the kth partial sum diverges. Consequenty the series also diverges.

- Using the improper integral of $f(x) = 1/x$ to show that the series diverges.

Conditions need to satisfy:
1. $f(x)$ is continuous over the interval $[1, \infty)$
2. $f(x)$ is decreasing
3. $f(n) = a_n$ for all integers $n \geq 1$.

the seq at n matches the function at n.



$$\sum_{n=1}^{k} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{k} > \int_1^{k+1} \frac{1}{x}\,dx$$

$k \to \infty$ $\quad$ $k \to \infty$

$$\lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{n} > \lim_{k\to\infty} \int_1^{k+1} \frac{1}{x}\,dx = \lim_{k\to\infty} \ln(x)\Big|_1^{k+1}$$

$$= \underbrace{\lim_{k\to\infty} \ln(k+1)}_{DNE} - \underbrace{\cancel{\lim_{k\to\infty} \ln(1)}}_{0}$$

Now, $\lim_{k\to\infty} \ln(k+1) \to DNE \to +\infty$.

So, this means that the kth partial sum is unbounded.

Thus, the kth partial sum of the harmonic series diverges.
Consequently, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Mini-Activity

Use the Integral test to determine whether the series below converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

b. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Reminder of the group assignments this week.

g1: Kizu C, Malory M, Jarod T
g2: Evan Ya, Taylor W, Jonathan N
g3: Nico C, Emma P, Evan Yo
g4: Nikaela R, Ronnie DS, Ryan O, Nathan PF

- Leaders Responsibility
  - make sure each member has some contribution.
  - make sure each member is heard.
  - make sure to keep track of the objective of the problem.