5.3 Integral Test Cont.

Monday, September 12, 2022

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Objectives:

1. Show some examples on:
   a. Evaluating standard integrals
   b. Evaluating integrals by substitution
   c. Integral test
2. Standard integrals and integral test worksheet

Groups for this week:
- g1: Nico C, Ronnie DS, Nathan PF
- g2: Kian C, Jarod T, Evan Yo.
- g3: Ryan O, Taylor W, Nikaela R
- g4: Evan Yo, Malory M, Emma P., Jonathan N.

Part I: Standard integrals

1. $$\int \frac{4}{\sqrt{4-x^2}} + \frac{2}{x^2+1}\,dx = 4\int \frac{1}{\sqrt{2^2-x^2}}\,dx + 2\int \frac{1}{x^2+1}\,dx$$

$$= 4\sin^{-1}\left(\frac{x}{2}\right) + 2\tan^{-1}(x) + C$$

2. $$\int \frac{1}{(x+2)^2} + \sin(x)\,dx = \int (x+2)^{-2}\,dx + \int \sin(x)\,dx$$

$$= -(x+2)^{-1} - \cos(x) + C$$

$$= -\left(\frac{1}{(x+2)} + \cos(x)\right) + C$$

Part II: Basic Integration by substitution

- Context: Derivatives by chain rule.

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\,g'(x) \rightarrow \text{ex. } f(x) = \frac{(x^2+1)^4}{4}$$

$$f'(x) = \frac{\cancel{4}(x^2+1)^3}{\cancel{4}}2x = 2x(x^2+1)^3$$

- Integration by substitution

$$\int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du$$

1. $$\int 2x(x^2+1)^3\,dx = \int \underbrace{\overbrace{(\underbrace{x^2+1}_{u(x)})^3}^{f(u(x))}}_{f(u)}\boxed{2x\,dx} \rightarrow du$$

$2x \downarrow \frac{du}{dx}$

let $u = x^2+1$

$du = 2x\,dx$

$\frac{du}{dx} = 2x$

$$= \int u^3\,du$$

$$= \frac{u^4}{4} + C \leftarrow \text{Put back } u = x^2+1$$

$$= \frac{(x^2+1)^4}{4} + C$$

$$= \frac{(x^2+1)^4}{4} + C$$

Put back $u = x^2+1$

2. $$\int \frac{3x^2}{x^3+1}\,dx = \int \frac{1}{x^3+1} 3x^2\,dx$$

let $u = x^3+1$
$du = 3x^2\,dx$

$$= \int \frac{1}{u}\,du$$
$$= \ln(u) + C$$
$$= \ln(x^3+1) + C$$

Part III: Integral Test

Given a series $\sum_{n=N}^{\infty} a_n$.

Step 1: Identify a_n.
Step 2: Write a $f(x)$ that match a_n.
Step 3: Check three conditions are satisfied
1. $f(x)$ is continuous over domain
2. $f(x)$ is decreasing & positive over domain
3. $f(x) = a_n$ for all $n \geq N$

Step 4: Make a conclusion about the series.

If $\int_N^\infty f(x)\,dx$ diverges, then $\sum_{n=N}^{\infty} a_n$ diverges.

If $\int_N^\infty f(x)\,dx$ converges, then $\sum_{n=N}^{\infty} a_n$ converges.

ex. 1. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Step 1: $a_n = 1/n^3$
Step 2: $f(x) = 1/x^3$
Step 3: $f(x)$ is decreasing & positive over domain

Step 4:
$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^3}\,dx$$
$$= \int_1^\infty x^{-3}\,dx$$
$$= \lim_{R\to\infty} \int_1^R x^{-3}\,dx$$

$$= \lim_{R\to\infty} \frac{x^{-2}}{-2}\Bigg|_1^R$$

$$= \cancel{\lim_{R\to\infty} -\frac{1}{2x^2}}^{\,0} + \lim_{R\to\infty} \frac{1}{2}$$

$$= 1/2$$

Since $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.