5.3 Integral Test Cont. & 5.4 Comparison Test

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Objectives:

1. Introducing the p-series
2. comparison test

Previously...

the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} \rightarrow \text{diverges}$$

You can show that the harmonic series diverges by (1) kth partial sum and (2) integral test.

Suppose we have the following series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \rightarrow \text{does this converge or diverge?}$$

the Integral test:

1. $a_n = 1/n^2$
2. $f(x) = 1/x^2$
3. $f(x)$ is continuous on the domain $[1,\infty)$.
   $f(x)$ is also decreasing and positive.
   $f(x) = a_n$ for all $n \geq 1$
4. $$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x^2}\,dx$$
   $$= \lim_{R\to\infty} -\frac{1}{x}\Big|_1^R$$
   $$= \cancel{\lim_{R\to\infty} -\frac{1}{R}}^{\,0} + \cancel{\lim_{R\to\infty} \frac{1}{1}}^{\,1}$$
   $$= 1 \rightarrow \text{the integral converges.}$$

Since $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges to 1,

then $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

then $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

The p-series

For any real number p, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is called the p-series.

1. If $p < 0$, then $\lim_{n\to\infty} \frac{1}{n^p} \to \infty$, DNE.
   By the divergence test $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p<0$.

2. If $p=0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 1$.
   By the divergence test $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p=0$.

3. If $p>0$, then $f(x) = 1/x^p$ is a positive, decreasing, and continuous function over domain $[1, \infty)$.
   We can use the integral test.

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x^p}\,dx$$

$$= \lim_{R\to\infty} \int_1^R x^{-p}\,dx \quad \text{use power rule}$$

$$= \lim_{R\to\infty} \left(\frac{1}{1-p}\right) x^{1-p} \Big|_1^R$$

$$= \lim_{R\to\infty} \left(\frac{1}{1-p}\right) R^{1-p} - \lim_{R\to\infty} \left(\frac{1}{1-p}\right) 1^{1-p}$$

$$= \frac{1}{1-p}\left(\lim_{R\to\infty} R^{1-p} - 1\right)$$

Cases: a. If $p<1$, then $\frac{1}{1-p}\left(\lim_{R\to\infty} R^{1-p} - 1\right) \to$ DNE.

positive value

positive

Cases: a. If $p < 1$, then $\frac{1}{1-p}\left(\lim_{R\to\infty} R^{1-p} - 1\right) \to$ DNE.

positive

So, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p < 1$.

b. If $p = 1$, then $\frac{1}{1-p}\left(\lim_{R\to\infty} R^{1-p} - 1\right) \to$ DNE

0 0

So, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p = 1$. $\to$ harmonic series.

negative value 0

c. If $p > 1$, then $\frac{1}{1-p}\left(\cancel{\lim_{R\to\infty} R^{1-p}} - 1\right) = \frac{-1}{1-p}$.

negative value

So, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

Therefore, the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \le 1 \end{cases}.$$

Comparison Test

This test is useful if the improper integral of the integral test is hard to evaluate or impossible to solve.

ex. 1. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3} \to \text{does this converge or diverge?}$$

a. Long way: Use the integral test.

- $a_n = 1/n^2+3$

- $f(x) = 1/x^2+3 \to$ positive, decreasing, continuous

$$\int_1^{\infty} \frac{1}{x^2+3}\,dx = \lim_{R\to\infty} \int_1^{\infty} \frac{1}{x^2+(\sqrt{3})^2}\,dx$$

$$= \lim \ \frac{1}{-} \tan^{-1}\left(\frac{x}{-}\right)\Big|^R$$

$$\int_1 \quad x^2+3 \qquad R\to\infty \int_1 \quad x^2+(\sqrt{3})^2$$

$$= \lim_{R\to\infty} \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\Big|_1^R$$

$$= \underbrace{\cancel{\lim_{R\to\infty} \frac{1}{\sqrt{3}} \tan^{-1}}\left(\frac{R}{\sqrt{3}}\right)}_{\pi/2\sqrt{3}} - \underbrace{\cancel{\lim_{R\to\infty} \frac{1}{\sqrt{3}} \tan^{-1}}\left(\frac{1}{\sqrt{3}}\right)}_{\pi/6\sqrt{3}}$$

$$= \frac{\pi}{3\sqrt{3}} \rightarrow \text{converges.}$$

By the Integral test, $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ converges.

b. short way: Use the comparison test.

comparing to something larger for all n.

We observe that $\underbrace{0 < \frac{1}{n^2+3}}_{\text{positive}} < \frac{1}{n^2}$ for $n \geq 1$.

The Kth partial sum of $\frac{1}{n^2+3}$ satisfies the inequality

$$S_K = \sum_{n=1}^{K} \frac{1}{n^2+3} < \sum_{n=1}^{K} \frac{1}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

p-series of $p=2$

Since $p=2$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

So, the kth partial sum S_K is bounded above

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ converges by the comparison test.

2. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1} \longrightarrow \text{does this converge or diverge?}$$

Observe that $0 < \frac{1}{2^n+1} < \frac{1}{2^n}$ for all $n \geq 1$

The Kth partial sum satisfies the inequality the kth partial sum satisfies the inequality

$$s_k = \sum_{n=1}^{k} \frac{1}{2^n+1} < \sum_{n=1}^{k} \frac{1}{2^n} < \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

geometric series with $r = 1/2$

Since $r < 1/2$, then $\sum_{n=1}^{\infty} r^n$ converges.

So, the kth partial sum is bounded above.

therefore, $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges by the comparison test.

Mini-Activity

Use the comparison test to determine if the following series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{1}{n - 1/3}$

2. $\sum_{n=1}^{\infty} \frac{3}{3^n+3}$

3. $\sum_{n=1}^{\infty} \frac{3}{n^2+n+3}$

4. $\sum_{n=1}^{\infty} \frac{n}{n^{3/2}+3}$