5.4 Comparison Test Cont.

Thursday, September 15, 2022

Quijano, AJ.

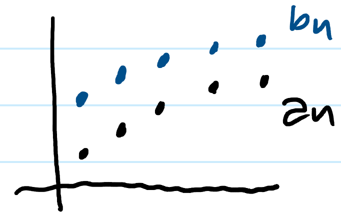
Objectives:

1. Introduce the limit comparison test.

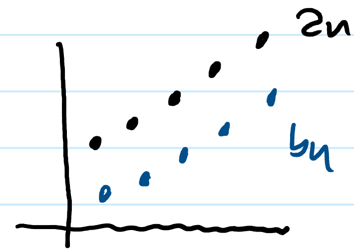
Previously...

<u>Comparison test</u>

1. Suppose $0 \leq a_n \leq b_n$ for all $n \geq N$.
   If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.



2. Suppose $a_n \geq b_n \geq 0$ for all $n \geq N$.
   If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.



ex. Use the comparison test to see if the following series converge or diverge.

1. $\sum_{n=1}^{\infty} \underbrace{\frac{2^n}{3^n+1}}_{a_n} \rightarrow$ Let $a_n = \frac{2^n}{3^n+1}$ and $b_n = \frac{2^n}{3^n}$.

So,
$$a_n \leq b_n$$
$$\frac{2^n}{^n+1} \leq \frac{2^n}{3^n}$$

$\rightarrow$ geometric series of the form $\left(\frac{2}{3}\right)^n$ with $r = 2/3$

Since the geometric series converges if $|r|<1$, then the series $\sum_{n=1}^{\infty} \frac{2^n}{3^n+1}$ also converges.

2. $\sum_{k=1}^{\infty} \underbrace{\frac{\ln(k)}{k}}_{a_n} \rightarrow$ Let $a_n = \frac{\ln(k)}{k}$ and $b_n = \frac{1}{k}$

So, $\quad a_n \geq b_n$

$$\frac{\ln(k)}{k} \geq \frac{1}{k}$$ → harmonic series which diverges

Since the harmonic series diverges, then the series $\displaystyle\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$ diverges.

Limit Comparison Test

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ be positive-termed series.

1. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is finite, and $c>0$, then either both series converge or both diverge.

2. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\displaystyle\sum_{n=n}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.
3. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

ex. 1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5n+10}$

Let's compare the series with a similar series, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. → We know the harmonic series diverges.

Let $a_n = \dfrac{1}{5n+10}$ and $b_n = \dfrac{1}{n}$.

So, $\dfrac{a_n}{b_n} = \dfrac{1/(5n+10)}{1/n} = \dfrac{n}{5n+10}$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{5n+10}$$

L'hopital → $\displaystyle = \lim_{n\to\infty} \frac{d}{d/dn[5n+10]}$

$\displaystyle = \lim_{n\to\infty} \frac{1}{5}$

$$= \lim_{n\to\infty} \frac{1}{5}$$

$$= \frac{1}{5} \rightarrow \text{positive \& non-}$$

So, both series behave the same. They diverge.
Therefore, $\sum_{n=1}^{\infty} \frac{1}{5n+10}$ diverges.

2. $\sum_{n=1}^{\infty} \frac{2^n}{2^n+3^n}$

We compare the series to a similar series when n is large.
So, $\frac{2^n}{2^n+3^n} \approx \frac{2^n}{3^n}$, when n is large.
Let's use $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ to compare which is a convergent series since $r<1$.

geometric series with $r = 2/3 < 1$

Let $a_n = \frac{2^n}{2^n+3^n}$ and $b_n = \left(\frac{2}{3}\right)^n$.

$$\text{So, } \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n/(2^n+3^n)}{(2/3)^n}$$

this is a technique called "multiplying 1"

$$= \lim_{n\to\infty} \frac{3^n}{2^n+3^n} \frac{(1/3^n)}{(1/3^n)}$$

$$= \lim_{n\to\infty} \frac{1}{\overset{0}{\cancel{(2^n/3^n)}}+1}$$

$$= 1 \rightarrow \text{positive \& non-zero}$$

So, both series behave the same. They both converge.
Therefore, $\sum_{n=1}^{\infty} \frac{2^n}{2^n+3^n}$ converges.

Mini-Activity

Use the limit comparison test to determine if the following series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n+2}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^4+2}$