5.5 Alternating Series

Monday, September 19, 2022

Objectives:

1. Introduce the alternating series.
2. Use the alternating series test.

Alternating Series

Definition: An alternating series is a series of the form

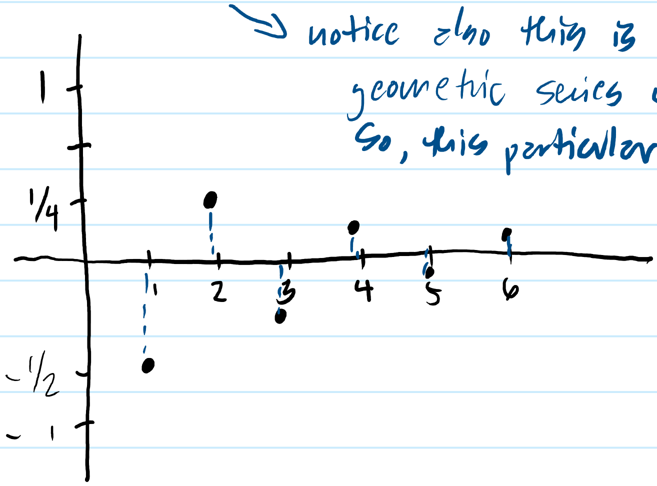
1. $\sum_{n=0}^{\infty} (-1)^n a_n$, → index start at $n=0$, or
2. $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, → index starts at $n=1$,

where $a_n > 0$ for each n.

ex.

1. $$\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n = -\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots$$

→ notice also this is a geometric series with $|r| < 1$. So, this particular alt. series converges.



2. $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

→ this is not a geometric series

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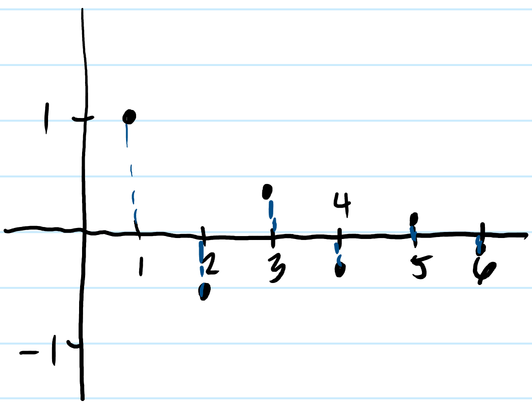
$$\sum_{n=1}^{\infty} (-1)^{n+1} \underbrace{\frac{1}{n}}_{a_n}$$

Here, $a_n = 1/n$.



Convergence of the Alternating Series

ex.

1. $\displaystyle\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n \rightarrow$ geometric series with $|r|<1$

Since $|r|<1$, then the series converges to

$$\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n = \frac{1}{1-(-1/2)} = \frac{1}{1+1/2} = \frac{1}{3/2} = \frac{2}{3}$$

2. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \underbrace{\frac{1}{n}}_{a_n}$

a. $a_n = 1/n$

b. $0 < a_{n+1} \leq a_n$ for all $n \geq 1$.
↳ positive sequence is monotonically decreasing

c. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n} \rightarrow 0$.

If b & c are true, then the alternating series converges

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Since b & c are true, then the series
$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \text{ converges.}$$

Alternating Series Test

An alternating series of the form
$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n} a_n$$

Converges if
1. $0 < a_{n+1} \leq a_n$ for all $n \geq 1$ and
2. $\lim_{n \to \infty} a_n = 0$.

ex.

3. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+2} = \sum_{n=1}^{\infty} \overset{\swarrow \text{the alternating term}}{(-1)^n} \underbrace{\frac{1}{n^2+2}}_{a_n}$$

a. $a_n = 1/n^2+2$
b. $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n^2+2} \to 0$

Therefore, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+2}$ converges.

Absolute convergence Definition

$\swarrow$ absolute value

1. If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. If $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

* Absolute ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms and

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \rho,$$ ← rho

i. If $\rho < 1$, the series converges absolutely.
ii. if $\rho > 1$, the series diverges
iii. if $\rho = 1$, then the test is inconclusive

ex.

4. $\sum_{n=1}^{\infty} (-1)^{n+1} \underbrace{\frac{n^2}{e^n}}_{a_n}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|(-1)^{n+2}(n+1)^2/e^{n+1}|}{|(-1)^{n+1}(n+1)^2/e^n|}$$

← terms $(-1)^n$ will cancel because of the absolute value.

$$= \lim_{n\to\infty} \frac{(n+1)^2}{e^1 n^2} = \frac{1}{e} \lim_{n\to\infty} \frac{(n+1)^2}{n^2}$$

L'hopital →

$$= \frac{1}{e} \lim_{n\to\infty} \frac{\cancel{2}(n+1)}{\cancel{2}n} = \frac{1}{e} \lim_{n\to\infty} \frac{1}{1}$$

$$= \frac{1}{e} < 1$$

Since $\rho < 1$, then the series converges absolutely.

Mini-Activity

Which series shown below converge and which diverge? which one converges absolutely?

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n}{n+5}$

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2. $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\ln(n)}$

3. $\sum_{n=1}^{\infty} (-2)^{2n}$

4. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+2}$