5.6 Ratio and Root Test

Wednesday, September 21, 2022

Objectives:

1. Revisit the absolute convergence definition.
2. Revisit the absolute ratio test.
3. Introduce the root test.

Absolute convergence Definition

absolute value

1. If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. If $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

Example:

1. consider the alternating harmonic series,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$ → it is alternating because of the $(-1)^n$ term

neg, pos, neg, pos

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots$$

$n=1$, $n=2$, $n=3$, $n \to 4$

Is the series above convergent?
Is it absolutely convergent or conditionally convergent?
→ Try the alternating series test.

a. $a_n = \frac{1}{n}$

b. $0 \le \underbrace{a_{n+1} \le a_n}$ for all $n \ge 1$.

$n=1$: $a_n = 1$

:

$\vdots$

$n = 3 : \; a_3 = \frac{1}{3}$

$n = 4 : \; a_4 = \frac{1}{4}$ $\quad \Big\} \; a_4 < a_3$

$n = 5 : \; a_5 = \frac{1}{5}$ $\quad \Big\} \; a_5 < a_4$

$\vdots$

a_n

a_{n+1} $\quad \Big\} \; a_{n+1} \leq a_n$

this condition is satisfied.

c. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} \to 0$. ✓

Thus, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent.

→ If you try the absolute ratio test, the test will be inconclusive because

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|1/n+1|}{|1/n|}$$

$$= \lim_{n\to\infty} \frac{|n|}{|n+1|}$$

L'hospital

$$= \lim_{n\to\infty} \frac{1}{1}$$

$$= 1.$$

→ Try the definition of absolute convergence.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$ } the harmonic series divergent

Since $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right|$ is divergent, then $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent.

Absolute Ratio Test (or the Ratio test)

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms and

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \rho.$$ ← rho

i. If $\rho < 1$, the series converges absolutely.
ii. if $\rho > 1$, the series diverges
iii. if $\rho = 1$, then the test is inconclusive

Note: Absolute convergence implies convergence.

ex. $\sum_{n=1}^{\infty} \underbrace{\frac{n^3}{3^n}}_{a_n}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|(n+1)^3/3^{n+1}|}{|n^3/3^n|}$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \frac{3^n}{n^3}$$

$$= \lim_{n\to\infty} \frac{\cancel{3^n}}{3^1 \cdot \cancel{3^n}} \left(\frac{(n+1)^3}{n^3}\right)$$

$$= \frac{1}{3} \lim_{n\to\infty} \frac{(n+1)^3}{n^3}$$

L'hospital →

$$= \frac{1}{3} \lim_{n\to\infty} \frac{\cancel{3}(n+1)^2}{\cancel{3}n^2}$$

L'hospital again →

$$= \frac{1}{3} \lim_{n\to\infty} \frac{\cancel{2}(n+1)}{\cancel{2}n}$$

$$= \frac{1}{3} \lim_{n\to\infty} \frac{1}{1}$$

$$\rho = \frac{1}{3} < 1$$

Since $\rho < 1$, then the series $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges absolutely.

Root Test

Consider the series $\sum_{n=1}^{\infty} a_n$.

nth root

Let $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$.

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

iii. If $\rho = 1$, the test is inconclusive.

Note the root test is useful if if you have a series with exponents with n.

ex. $\sum_{n=1}^{\infty} (-1)^n \underbrace{\left(\frac{3n-1}{4n+2}\right)^{n}}_{a_n}$ exponent as the variable n

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{3n-1}{4n+2}\right)^n\right|}$$

$$= \lim_{n\to\infty} \left|\left(\frac{3n-1}{4n+2}\right)^{\cancel{n/n}\,1}\right|$$

$$= \lim_{n\to\infty} \frac{3n-1}{4n+2} \frac{(1/n)}{(1/n)}$$

here, i am applying the "multiplying by 1" method

$$= \lim_{n\to\infty} \frac{3-\cancel{1/n}^{\,0}}{4+\cancel{2/n}^{\,0}}$$

$$\rho = \frac{3}{4} < 1$$

Since $\rho < 1$, then the series $\sum_{n=1}^{\infty} (-1)^n \left(\frac{3n-1}{4n+2}\right)^n$ converges absolutely.

Useful Limits to know

purpose: to aid you if you need to use the root test.

1. $\lim_{n\to\infty} \sqrt[n]{n^a} = \lim_{n\to\infty} n^{a/n} \rightarrow$ may look like an indeterminate form ∞^0

but actually for any positive number a,

$$\lim_{n\to\infty} \sqrt[n]{n^a} = \lim_{n\to\infty} n^{a/n} = 1 \quad \text{and}$$

$$\lim_{n\to-\infty} \sqrt[n]{n^a} = \lim_{n\to-\infty} n^{a/n} = 1.$$

Mini-Activity

Use the ratio or root test to determine if the following series converges? absolutely convergent or conditionally convergent?

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^n}$
2. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{3}{2n^2}\right)^n$
3. $\sum_{n=1}^{\infty} (-1)^n \frac{n^4+4}{4^n}$
4. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{3n+2}{2n^2+3}\right)^n$