1.5 Substitution

Monday, September 26, 2022

Objectives:

1. Revisit the standard integration guide.
2. Revisit the basic u-substitution technique.
3. Introduce some intermediate u-substitutions.

Derivatives by chain rule

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\, g'(x)$$

ex. $f(x) = \dfrac{(x^2+1)^4}{4}$

$$f'(x) = \frac{\cancel{4}(x^2+1)^3}{\cancel{4}} 2x = 2x(x^2+1)^3$$

Basic Integration by substitution (aka change of variable)

Let $u(x) = g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

$$= F(u) + C$$

$$= F(g(x)) + C$$

1. $$\int 2x(x^2+1)^3\, dx = \int \overbrace{(\underbrace{x^2+1}_{u=g(x)})^3}^{f(u)} \boxed{\underbrace{2x}_{g'(x)}\, dx} \leftarrow du$$

let $u = x^2+1$

$\dfrac{du}{dx} = 2x$

$du = 2x\,dx$

$$= \int u^3\, du \longleftarrow \text{use power rule}$$

$$= \frac{u^4}{4} + C \longleftarrow \text{Put back } u = x^2+1$$

$$du = 2x\,dx \qquad = \frac{u^4}{4} + C$$

← Put back $u = x^2+1$

$$= \frac{(x^2+1)^4}{4} + C$$

$$\int 2x(x^2+1)^3\,dx = \frac{(x^2+1)^4}{4} + C$$

2. $$\int \frac{3x^2}{x^3+1}\,dx = \int \frac{1}{x^3+1}\,3x^2\,dx$$

let $u = x^3+1$

$du = 3x^2\,dx$

$$= \int \frac{1}{u}\,du$$ ← standard integral of $\frac{1}{x}$.

$$= \ln(u) + C$$

$$= \ln(x^3+1) + C$$

$$\int \frac{3x^2}{x^3+1}\,dx = \ln(x^3+1) + C$$

check: let $F(x) = \ln(x^3+1) + C$ ← chain rule

$$F'(x) = f(x) = 3x^2\left(\frac{1}{x^3+1}\right)$$

$$f(x) = \frac{3x^2}{x^3+1} \quad ✓$$

3. $$\int 6x(3x^2+4)^4\,dx = \int \overbrace{(\underbrace{3x^2+4}_{g(x)})^4}^{f(u)}\ \boxed{\underbrace{6x}_{g'(x)}\,dx} \to du$$

let $u = 3x^2+4$

$\frac{du}{dx} = 6x$

$du = 6x\,du$

$$= \int u^4\,du$$ ← use power rule

$$= \frac{u^5}{5} + C$$ ← put back $u = 3x^2+4$

$$= \frac{(3x^2+4)^5}{5} + C$$

$$\int 6x(3x^2+4)^4\,dx = \frac{(3x^2+4)^5}{5} + C$$

Check: Let $F(x) = \frac{(3x^2+4)^5}{5} + C$

$$F'(x) = f(x) = \frac{5(3x^2+4)^4}{5}(6x) \leftarrow \text{chain rule}$$

$$f(x) = (3x^2+4)^4\,6x \quad \checkmark$$

4. $$\int \frac{\cos(x)}{\sin^3(x)}\,dx = \int \overbrace{\frac{1}{\underbrace{(\sin(x))}_{g(x)}{}^3}}^{f(u)} \underbrace{\boxed{\cos(x)\,dx}}_{g'(x)} \leftarrow du$$

let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$

$du = \cos(x)\,dx$

$$= \int \frac{1}{u^3}\,du \leftarrow \text{power rule}$$

$$= -\frac{1}{2u^2}$$

$$= -\frac{1}{2(\sin(x))^2} + C = -\frac{1}{2}\csc^2(x) + C$$

$$\int \frac{\cos(x)}{\sin^3(x)}\,dx = -\frac{1}{2}\csc^2(x) + C$$

Check: Let $F(x) = \frac{1}{2}\csc^2(x) + C$ $\leftarrow$ chain rule

$$F'(x) = f(x) = \frac{\cancel{2}}{\cancel{2}}\csc(x)(\cot(x)\csc(x))$$

$$= \cot(x)\csc^2(x)$$

$$= \left(\frac{\cos(x)}{\sin(x)}\right)\left(\frac{1}{\sin^2(x)}\right)$$

$$f(x) = \frac{\cos(x)}{\sin^3(x)} \quad \checkmark$$

Mini-Activity

Evaluate the following indefinite integrals, and

Evaluate the following indefinite integrals, and check your answers.

a. $\int (x-1)^5\, dx$

b. $\int (2x+5)(x^2+5x)^7\, dx$

c. $\int \cos(x)\sin^2(x)\, dx$

d. $\int \frac{2x}{\sqrt{x^2+1}}\, dx$

Intermediate Integration by Substitution

u-substitution with alteration.

5. $\int x\sqrt{x^2-5}\, dx = \int \sqrt{x^2-5}\, \boxed{x\, dx}$

let $u = x^2-5$

$\frac{du}{dx} = 2x$

$du = 2x\, dx$

$\boxed{\frac{1}{2} du = x\, dx}$

$= \int \sqrt{u}\, \frac{1}{2}\, du$

$= \frac{1}{2}\int \sqrt{u}\, du$ ← power rule

$= \frac{1}{2} \frac{u^{3/2}}{3/2} + C$

$= \left(\frac{1}{\cancel{2}}\right)\left(\frac{\cancel{2}}{3}\right)(x^2-5)^{3/2} + C$

$\int x\sqrt{x^2-5}\, dx = \frac{(x^2-5)^{3/2}}{3} + C$

Check: let $F(x) = (x^2-5)^{3/2} + C$ ← chain rule

Check: Let $F(x) = \frac{(x^2-5)^{3/2}}{3} + C$ ← chain rule

$$F'(x) = f(x) = \left(\frac{1}{\cancel{3}}\right)\left(\frac{\cancel{3}}{\cancel{2}}\right)(x^2+5)^{1/2}(\cancel{2}x)$$

$$= \sqrt{x^2+5}\; x \quad \checkmark$$

6. $\int (3-x)^4 dx = \int (3-x)^4 \boxed{dx}$

Let $u = 3-x$

$\frac{du}{dx} = (-1)$

$du = (-1)\,dx$

$\boxed{(-1)\,du = dx}$

$$= \int u^4 (-1)\, du$$

$$= -\int u^4 du$$ ← power rule

$$= -\frac{u^5}{5} + C$$

$$= -\frac{(3-x)^5}{5} + C$$

$$\int (3-x)^4 dx = -\frac{(3-x)^5}{5} + C$$

Check: Let $F(x) = -\frac{(3-x)^5}{5} + C$ ← chain rule

$$F'(x) = f(x) = \cancel{-}\frac{\cancel{5}(3-x)^4}{\cancel{5}}(\cancel{-}1)$$

$$f(x) = (3-x)^4 \quad \checkmark$$

Mini-Activity

Evaluate the following indefinite integrals, and check your answers.

e. $\int \sqrt{2x+3}\, dx$

f. $\int \frac{\sin(x)}{\;}\, dx$

f. $\int \frac{\sin(x)}{\cos^3(x)}\,dx$

g. $\int \frac{x}{\sqrt{1-x^2}}\,dx$

h. $\int \cos^3(x)\sin(x)\,dx$