1.5 Substitution Cont.

Wednesday, September 28, 2022

Objectives:

1. Continue Integration by substitution.
   a. exponential functions
   b. logarithmic functions
3. Problem solving strategy for Integration by substitution.
2. Integration by substitution for definite integrals.

Integrals of exponential & logarithmic functions

- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\ln(a)} + C$ for $a \neq 1$ & $a > 0$
- $\int \frac{1}{x} dx = \ln(|x|) + C$
- $\int \ln(x)\, dx = x \ln x - x + C$
- $\int \log_a x\, dx = \frac{x}{\ln(a)}(\ln(x) - 1) + C$ for $a > 0$ & $a \neq 0$

Note that a^x is called an exponential because the vaiable is in the exponent with constant base.
A polynomial is when you have a variable base and a constant exponent.

ex. $2^x \rightarrow$ exponential
$x^2 \rightarrow$ polynomial

Applying Integration by Substitution on exponentials

1. $$\int x^2 e^{-2x^3} dx = \int \overbrace{e^{\underbrace{-2x^3}_{u}}}^{f(u)} \underbrace{\boxed{x^2\, dx}}_{du/dx}$$

let $u = -2x^3$

$\frac{du}{dx} = -6x^2$

$du = -6x^2 dx$

$-\frac{1}{6} du = \boxed{x^2 dx}$

$$= \int e^u \left(-\frac{1}{6}\right) du$$ substitution with alteration

$$= -\frac{1}{6} \int e^u du$$

$$= -\frac{1}{6} e^u + C$$

put back $u = -2x^3$

$$\int x^2 e^{-2x^3} dx = -\frac{1}{6} e^{-2x^3} + C$$

check: let $F(x) = -\frac{1}{6} e^{-2x^3} + C$

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$$F'(x) = f(x) = \left(\cancel{-}\frac{1}{\cancel{6}} e^{-2x^3}\right)(\cancel{-6}x^2)$$

$$= e^{-2x^3} x^2 \quad \checkmark$$

2. $\displaystyle\int e^x \sqrt{1+e^x}\, dx = \int \sqrt{1+e^x}\, e^x dx$

let $u = 1+e^x$

$\frac{du}{dx} = e^x$

$du = e^x dx$

$$= \int \sqrt{u}\, du \quad \leftarrow \text{no alterations}$$

$$= \int u^{1/2} du$$

$$= \frac{u^{3/2}}{3/2} + C$$

$$\int e^x \sqrt{1+e^x}\, dx = \frac{(1+e^x)^{3/2}}{3/2} + C$$

$$= \frac{2}{3}(1+e^x)^{3/2} + C$$

check: let $F(x) = \frac{2}{3}(1+e^x)^{3/2} + C$

$$F'(x) = f(x) = \left(\frac{\cancel{3}}{\cancel{2}}\right)\left(\frac{\cancel{2}}{\cancel{3}}\right)(1+e^x)^{3/2-1} e^x$$

$$= (1+e^x)^{1/2} e^x$$

Mini-Activity

Evaluate the following integrals.

a. $\displaystyle\int \sqrt{e^x}\, e^x\, dx$

b. $\displaystyle\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\, dx$

c. $\displaystyle\int x 3^{x^2}\, dx$

d. $\displaystyle\int \frac{3^{1/x^2}}{x^3}\, dx$

Integration by Substitution for Definite Integrals

3. $\displaystyle\int_1^2 (4x^2+4)^3 \boxed{2x\, dx} = \frac{1}{4}\int_8^{20} u^3 du$

$$\int_1 \quad (4x \; \boxed{\ldots dx} \qquad \frac{1}{4}\int_8 \quad u \, du$$

Let $u = 4x^2+4$

$\frac{du}{dx} = 8x$

$du = 8x\,dx$

$\frac{1}{4}du = \boxed{2x\,dx}$

Bounds: $x=1 \to u(1) = 4(1)^2+4 = 8$

$x=2 \to u(2) = 4(2)^2+4 = 20$

↳ this works as long as the u is continuous over the interval.

$$= \frac{1}{4}\left[\frac{u^4}{4}\right]\Big|_8^{20}$$

In this case we don't have to put back u.

$$= \frac{(20)^4}{16} - \frac{(8)^4}{16}$$

$$= 9744$$

4. $$\int_1^2 e^{1-x}\,dx = -\int_0^{-1} e^u\,du$$

notice the negative sign

Let $u = 1-x$

$\frac{du}{dx} = -1$

$-du = dx$

bounds: $x=1 \to u(1) = 1-1 = 0$

$x=2 \to u(2) = 1-2 = -1$

$$= \int_{-1}^{0} e^u\,du$$

becomes positive if bounds are switched

$$= e^u\Big|_{-1}^{0}$$

$$\int_1^2 e^{1-x}\,dx = e^0 - e^{-1}$$

$$= 1 - \frac{1}{e}$$

Mini-Activity

Evaluate the following integrals.

e. $\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

f. $\int_1^2 \frac{1}{x^3} e^{4x^{-2}}\,dx$

g. $\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$

h. $\int_0^9 \sqrt{4-\sqrt{x}}\,dx$

Problem-Solving Strategy for Integration by Substitution

1. choose a $u=g(x)$ such that $du/dx = g'(x)$ is part of the integrand.
   - think about composite functions.
2. If your chosen $u=g(x)$ ends up a complicated integral in terms of u, then choose another u.
3. If all else fails, move on to other substitution or integration techniques. (more discussion on other integration techniques starting next week)