3.1 Integration By Parts

Friday, September 30, 2022

Objectives:

1. Introduce the method of Integration by parts
2. Work on integration by parts example problems

Previously...

Derivative by chain rule

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$$

Integration by substitution

let $u = g(x)$.

$$\int f(u(x)) \frac{du}{dx}\, dx = \int f(u)\, du$$

Derivative by Product rule

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

↓ say we integrate both sides

$$\int \frac{d}{dx} f(x)g(x)\, dx = \int f'(x)g(x)\, dx + \int f(x)g'(x)\, dx$$

$$f(x)g(x) = \int f'(x)g(x)\, dx + \underbrace{\int f(x)g'(x)\, dx}$$

we want this term

$$\int f(x)\,g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Let $u = f(x)$ and $v = g(x)$

So, $\frac{du}{dx} = f'(x)$ and $\frac{dv}{dx} = g'(x)$

$du = f'(x)\,dx$ $\quad$ $dv = g'(x)\,dx$

$$\int u\,dv = uv - \int v\,du$$

Integration by parts

Let $u = f(x)$ and $v = g(x)$ with continuous derivatives.

then, $\int u\,dv = uv - \int v\,du$.

Examples:

1. $\int x\sin(x)\,dx$

Let's try a normal u-substitution.

$$\int x\sin(x)\,dx = \int \; ?$$

let $u = \sin(x)$

$du = \cos(x)\,dx$

$\overset{?}{0}$

It does not work.

Let's try integration by parts.

$$\int x\sin(x)\,dx = \int \underset{f(x)}{x}\ \underset{g'(x)\,dx}{\sin(x)\,dx}$$

Let $u = x$ and $dv = \sin(x)\,dx$

So, $\frac{du}{dx} = 1$ and $v = \int \sin(x)\,dx$

$du = dx$ $\quad v = -\cos(x)$

$$\int x\sin(x)\,dx = x(-\cos(x)) - \int (-\cos(x))\,dx$$

$$= -x\cos(x) + \int \cos(x)\,dx$$

$$\int x\sin(x)\,dx = -x\cos(x) + \sin(x) + C$$

What if you choose $u = \sin(x)$ & $dv = x$? You will discover that you end up a more complicated integral.

2. $$\int \frac{\ln x}{x^3}\,dx = \int \ln(x)\,x^{-3}\,dx$$

Let $u = \ln(x)$ and $dv = x^{-3}\,dx$

$\frac{du}{dx} = \frac{1}{x}$ $\qquad v = \int x^{-3}\,dx$

$v = -\frac{1}{2}x^{-2}$

dx $\times$ v

$du = x^{-1}dx$ $\quad v = -\frac{1}{2}x^{-2}$

$$\int \ln(x)\,x^{-3}dx = \ln(x)\left(-\frac{1}{2}x^{-2}\right) - \int \left(-\frac{1}{2}x^{-2}\right)x^{-1}dx$$

$$= -\frac{\ln(x)}{2x^2} + \frac{1}{2}\int \frac{1}{x^3}\,dx$$

$$= -\frac{\ln(x)}{2x^2} + \frac{1}{2}\left(-\frac{1}{2x^2}\right) + C$$

$$= -\frac{\ln(x)}{2x^2} - \frac{1}{4x^2} + C$$

Group example problems

Evaluate the following integrals using integration by parts

For each group discuss the following
i. what u and dv to choose?
ii. what strategies did you use to choose u and dv?
iii. what is the final solution? check your answers.

a. $\int xe^{2x}dx$

b. $\int x\ln(x)\,dx$

c. $\int x^2e^x\,dx$

c. $\int x^2 e^x \, dx$

d. $\int x \sin(x) \log(x) \, dx$