3.1 Integration by Parts Cont.

Monday, October 3, 2022

Objectives:
1. Continue practicing Integration by parts
2. Definite integrals using Integration by parts

Integration by parts (IBP)

Let $u=f(x)$ and $v=g(x)$ be differentiable functions. then

$$\int u\,dv = uv - \int v\,du.$$

Examples:

1. $\int x^2 e^x dx \longrightarrow$ IBP twice

$\int u\,dv = uv - \int v\,du$

$\int x^2 e^x dx = x^2 e^x - \int e^x 2x\,dx$

$\int w\,dz$

1st. Let $u=x^2$ and $dv = e^x dx$

$du = 2x\,dx$ $\quad v = \int e^x dx$

$v = e^x$

2nd. Let $w = 2x$ and $dz = e^x dx$

$dw = 2\,dx$ $\quad z = \int e^x dx$

$z = e^x$

$\int w\,dz = wz - \int z\,dw$

$$\int x^2 e^x dx = x^2 e^x - \left(2xe^x - \int e^x 2\,dx\right)$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

$$= e^x(x^2 - 2x + 2)$$

2. $\int x\sin(x)\cos(x)\,dx$

Let $u = x$ and $dv = \sin(x)\cos(x)\,dx$

$du = dx$ $\quad v = \int \sin(x)\cos(x)\,dx \leftarrow$ use u-substitution

$v = \frac{1}{2}\sin^2(x)$

$\int u\,dv = uv - \int v\,du$

use trig identity

$\cos(2x) = 1 - 2\sin^2(x)$

$\sin^2(x) = \frac{1}{2}(1-\cos(2x))$

$$\int x\sin(x)\cos(x)\,dx = x\left(\frac{1}{2}\sin^2(x)\right) - \int \frac{1}{2}\sin^2(x)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

use the identity

$\cos(2x) = 1 - 2\sin^2(x)$

$\sin^2(x) = \frac{1}{2}(1-\cos(2x))$

$$\int x\sin(x)\cos(x)\,dx = x\left(\frac{1}{2}\sin^2(x)\right) - \int \frac{1}{2}\sin^2(x)\,dx$$

$$= \frac{x}{2}\sin^2(x) - \frac{1}{2}\int \frac{1}{2}(1-\cos(2x))\,dx$$

use u-substitution again

$$= \frac{x}{2}\sin^2(x) - \frac{1}{4}\int (1-\cos(2x))\,dx$$

$$= \frac{x}{2}\sin^2 x - \frac{1}{4}\left(x - \frac{\sin(2x)}{2}\right) + C$$

$$= \frac{x}{2}\sin^2 x - \frac{x}{4} + \frac{1}{8}\sin(2x) + C$$

3. $\int e^x\cos(x)\,dx \longrightarrow$ IBP twice and a little algebra

Let $u = \cos(x)$ and $dv = e^x\,dx$

$du = -\sin(x)$ $\quad v = \int e^x\,dx$

$v = e^x$

$$\int u\,dv = uv - \int v\,du$$

$$\int e^x\cos(x)\,dx = \cos(x)e^x - \int e^x(-\sin(x))\,dx$$

$$= \cos(x)e^x + \int e^x\sin(x)\,dx$$

Let $w = \sin(x)$ and $dz = e^x\,dx$

$dw = \cos(x)$ $\quad z = e^x$

$$\int w\,dz = wz - \int z\,dw$$

$$\int e^x\sin(x)\,dx = \sin(x)e^x - \int e^x\cos(x)\,dx$$

$$\int e^x\cos(x)\,dx = \cos(x)e^x + \sin(x)e^x - \int e^x\cos(x)\,dx$$

$+\int e^x\cos(x)\,dx$ $\qquad\qquad +\int e^x\cos(x)\,dx$

$$2\int e^x\cos(x)\,dx = e^x\cos(x) + e^x\sin(x)$$

$$\int e^x\cos(x)\,dx = \frac{e^x}{2}(\cos(x)+\sin(x)) + C$$

IBP for Definite Integrals

Let $u = f(x)$ and $v = g(x)$ be differential functions.
Then,

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Example:

4. $\int_0^1 \tan^{-1}(x)\,dx$

Let $u = \tan^{-1}(x)$ and $dv = dx$

$du = \frac{1}{1+x^2}dx \qquad v = \int dx$

$v = x$

$$\int u\,dv = uv - \int v\,du$$

$$\int_0^1 \tan^{-1}(x)\,dx = \tan^{-1}(x)x\Big|_0^1 - \int_0^1 x\frac{1}{1+x^2}dx$$ ← use u-substitution

Let $u = 1+x^2$ and $du = 2x\,dx$

$$= x\tan^{-1}(x)\Big|_0^1 - \frac{1}{2}\ln(|x^2+1|)\Big|_0^1$$

$$= \tan^{-1}(1) - \underbrace{\cancel{0(\tan^{-1}(0))}}_{0} - \left(\frac{1}{2}\ln(2) - \underbrace{\cancel{\frac{1}{2}\ln(1)}}_{0}\right)$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln(2)$$

<u>Mini-Activity</u>

Evaluate the following integrals using IBP.

a. $\int \left(\frac{\ln(x)}{x}\right)^2 dx \longrightarrow u = (\ln(x))^2$ and $dv = 1/x^2\,dx$, $w = \ln(x)$ and $dz = -1/x$

b. $\int_6^0 (2+5x)e^{x/3}\,dx \longrightarrow u = 2+5x$ and $dv = e^{x/3}\,dx$

c. $\int_0^\pi x^2\cos(4x)\,dx \longrightarrow u = x^2$ and $dv = \cos(4x)\,dx$, $w = x$ and $dv = \sin(4x)\,dx$

d. $\int e^{2x}\cos\left(\frac{1}{4}x\right)dx \longrightarrow u = \cos\left(\frac{1}{4}x\right)$ and $dv = e^{2x}\,dx$, $w = \sin\left(\frac{1}{4}x\right)$ and $dv = e^{2x}\,dx$

This needs a little bit of algebraic manipulation.