3.2 Trigonometric Integrals

Wednesday, October 5, 2022

Objectives:

1. Products and powers of $\sin(x)$ and $\cos(x)$ integrals.

<u>Products and powers of $\sin(x)$ and $\cos(x)$</u>

Integrals of the form:

$$\int \cos^j(x)\sin^k(x)\,dx.$$

Strategies:

1. If k is an odd number, rewrite $\sin^k(x) = \sin^{k-1}(x)\sin(x)$
   and use the identity $\sin^2(x) = 1-\cos^2(x)$
   to rewrite $\sin^{k-1}(x)$ in terms of $\cos(x)$.
   Integrate using u-substitution $u = \cos(x)$
   $du = -\sin(x)\,dx$.
2. If j is an odd number, rewrite $\cos^j(x) = \cos^{j-1}(x)\cos(x)$
   and use the identity $\cos^2(x) = 1-\sin^2(x)$
   to rewrite $\cos^{j-1}(x)$ in terms of $\sin(x)$.
   Integrate using u-substitution $u = \sin(x)$
   $du = \cos(x)\,dx$.
3. If both j and k are odd numbers, then use 1 or 2.
   d. $\int \tan^5(x)\sec^2(x)\,dx$
4. If both j and k are even numbers, use $\sin^2(x) = (1/2) - (1/2)\cos(2x)$
   and $\cos^2(x) = (1/2) + (1/2)\cos(2x)$.
   After simplifying, use 1 or 2.

Examples:

- $\int \cos^2(x)\sin^3(x)\,dx$ (j even, k odd) $= \int \cos^2(x)\sin^2(x)\sin(x)\,dx$ ($k-1$)

  use identity $\sin^2(x) = 1-\cos^2(x)$

  $= \int \cos^2(x)\left(1-\cos^2(x)\right)\sin(x)\,dx$

  use u-substitution

$$= \int \cos(x)\,(1-\cos(x))\,\sin(x)\,dx$$

use u-substitution

Let $u = \cos(x)$ and $du = -\sin(x)\,dx$

$$= -\int u^2 (1-u^2)\,du$$

$$= \int (u^4 - u^2)\,du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

→ put back $u = \cos(x)$

$$= \frac{\cos^2(x)}{5} - \frac{\cos^3(x)}{3} + C$$

← even

- $$\int \sin^4(x)\,dx = \int (\sin^2(x))^2\,dx$$

use identity $\sin^2(x) = (1/2) - (1/2)\cos(2x)$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2 dx$$

→ even

$$= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(x)\right) dx$$

← doubled coeff

use identity $\cos^2(2x) = (1/2) + (1/2)\cos(4x)$

$$= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right)\right) dx$$

$$= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{8} + \frac{1}{8}\cos(4x)\right) dx$$

$$= \int \left(\frac{3}{8} - \frac{1}{2}\cos(2) + \frac{1}{8}\cos(4x)\right) dx$$

$$= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$$

Integrating products of sines and cosines of different angles.

Trigonometric Transformations

- $\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$
- $\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$

- $\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$

Example:

- $\int \sin(5x)\cos(3x)\,dx = \int \frac{1}{2}\sin((5-3)x) + \frac{1}{2}\sin((5+3)x)\,dx$

$$= \int \frac{1}{2}\sin(2x) + \frac{1}{2}\sin(8x)\,dx$$

$$= -\frac{1}{4}\cos(2x) - \frac{1}{16}\cos(8x) + C$$

Mini-Activities

Evaluate the following integrals.

a. $\int \cos^3(x)\sin^2(x)\,dx$

b. $\int \cos^3(x)\,dx$

c. $\int \cos(6x)\cos(5x)\,dx$