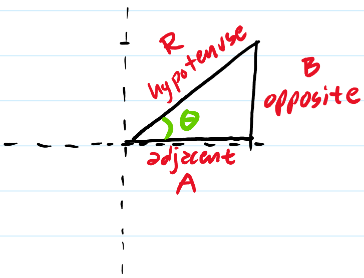
3.3 Trigonometric Substitution

Monday, October 10, 2022

Objectives

1. Introduce Trigonometric substitution

Pythagorean theorem



$$A = \sqrt{R^2 - B^2}$$
$$B = \sqrt{R^2 - A^2}$$
$$R = \sqrt{A^2 + B^2}$$

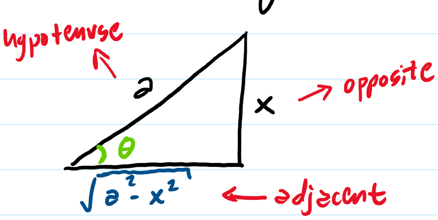
In terms of θ

$$R\sin(\theta) = B$$
$$R\cos(\theta) = A$$
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{B/C}{A/C} = \frac{B}{A}$$

$$A^2 + B^2 = R^2$$
$$R^2\cos^2(\theta) + R^2\sin^2(\theta) = R^2$$
$$\cos^2(\theta) + \sin^2(\theta) = R^2/R^2$$
$$\cos^2(\theta) + \sin^2(\theta) = 1$$

Reference Triangles

1. Integrals involving $\sqrt{a^2 - x^2}$.

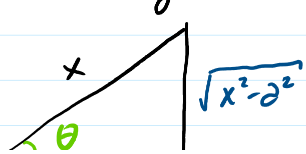


$$\sin(\theta) = \frac{x}{a} \longrightarrow \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta) = \frac{\sqrt{a^2 - x^2}}{a} \longrightarrow \frac{\text{adjacent}}{\text{hypotenuse}}$$

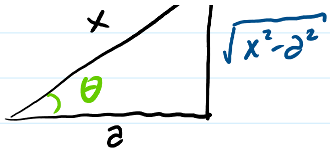
$$\tan(\theta) = \frac{x}{\sqrt{a^2 - x^2}} \longrightarrow \frac{\text{opposite}}{\text{adjacent}}$$

2. Integrals involving $\sqrt{x^2 - a^2}$.



$$\sin(\theta) = \frac{\sqrt{x^2 - a^2}}{x}$$

$\cos(\theta) = a$

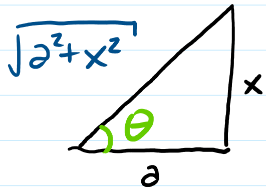


$$\sin(\theta) = \frac{\sqrt{x^2-a^2}}{x}$$

$$\cos(\theta) = \frac{a}{x}$$

$$\tan(\theta) = \frac{\sqrt{x^2-a^2}}{a}$$

3. Integrals involving $\sqrt{a^2+x^2}$.



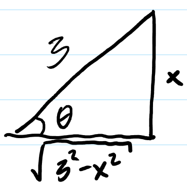
$$\sin(\theta) = \frac{x}{\sqrt{a^2+x^2}}$$

$$\cos(\theta) = \frac{a}{\sqrt{a^2+x^2}}$$

$$\tan(\theta) = \frac{x}{a}$$

Example

- $\int \sqrt{9-x^2}\,dx = \int \sqrt{3^2-x^2}\,dx$



$\sin(\theta) = \frac{x}{3}$

$x = 3\sin(\theta)$

$\frac{dx}{d\theta} = 3\cos(\theta)$

$dx = 3\cos(\theta)d\theta$

$\theta = \sin^{-1}\left(\frac{x}{3}\right)$

$\sin(\theta) = \frac{x}{3}$

$\cos(\theta) = \frac{\sqrt{3^2-x^2}}{3}$

$$= \int \sqrt{3^2-(3\sin^2\theta)^2}\; 3\cos(\theta)d\theta$$

$$= \int \sqrt{3^2(1-\sin^2\theta)}\; 3\cos(\theta)d\theta$$

Apply $\cos^2(\theta) = 1-\sin^2(\theta)$

$$= \int \sqrt{3^2\cos^2(\theta)}\; 3\cos(\theta)d\theta$$

$$= \int 9\cos^2(\theta)d\theta$$

even

trigonometric integral

Apply $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$

$$= \int 9\left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)d\theta$$

$$= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C$$

Apply $\sin(2\theta) = 2\sin\theta\cos(\theta)$

$$= \frac{9}{2}\theta + \frac{9}{4}(2\sin\theta\cos\theta) + C$$

Apply $\sin(2\theta) = 2\sin\theta\cos\theta$

$$= \frac{9}{2}\theta + \frac{9}{4}(2\sin\theta\cos\theta) + C$$

Substitute back the terms

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$\int \sqrt{9-x^2}\,dx = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x}{2}\sqrt{9-x^2} + C$$

Mini-Activity

For the following integrals

i. draw the reference triangle

ii. Apply the trigonometric substitution

iii. Evaluate the integral.

1. $\int \frac{\sqrt{4-x^2}}{x}\,dx \longrightarrow \sin(\theta) = \frac{x}{2}, \; x = 2\sin(\theta), \; dx = 2\cos(\theta)\,d\theta$

2. $\int \frac{x^3}{\sqrt{25-x^2}}\,dx \longrightarrow \sin(\theta) = \frac{x}{5}, \; x = 5\sin(\theta), \; dx = 5\cos(\theta)\,d\theta$

3. $\int x^3\sqrt{x^2+4}\,dx \longrightarrow \tan(\theta) = \frac{x}{2}, \; x = 2\tan(\theta), \; dx = 2\sec^2(\theta)\,d\theta$

4. $\int \sqrt{x^2-9}\,dx \longrightarrow \cos(\theta) = \frac{3}{x}, \; x = \frac{3}{\cos(\theta)} = 3\sec(\theta), \; dx = 3\tan(\theta)\sec(\theta)\,d\theta$