3.3 Trigonometric Substitution Cont.

Wednesday, October 12, 2022

Objectives:

1. Continue Trigonometric Substitution
2. Using reduction formulas to solve trigonometric integrals.

Reduction Formulas

- $\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$
- $\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx$

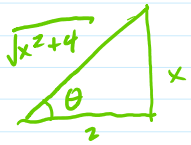
Examples:

- $\int \tan^3(x)\,dx$ ($n=3$)
$$= \frac{1}{3-1}\tan^{3-1}(x) - \int \tan^{3-2}(x)\,dx$$
$$= \frac{1}{2}\tan^2(x) - \int \tan(x)\,dx$$
$$= \frac{1}{2}\tan^2(x) - \left(-\ln(|\cos(x)|)\right) + C$$
$$= \frac{1}{2}\tan^2(x) + \ln(|\cos(x)|) + C$$

- $\int \sec^3(x)\,dx$ ($n=3$)
$$= \frac{1}{3-1}\sec^{3-1}(x)\tan(x) + \frac{3-2}{3-1}\int \sec^{3-2}(x)\,dx$$
$$= \frac{1}{2}\sec^2(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx$$
$$= \frac{1}{2}\sec^2(x)\tan(x) + \frac{1}{2}\left(\ln|\sec(x)+\tan(x)|\right) + C$$

Trigonometric Substitution Examples

- $\int x^3\sqrt{x^2+4}\,dx$



$\sec(\theta) = \frac{1}{\cos(\theta)}$

$\sec(\theta) = \frac{1}{2/\sqrt{x^2+4}}$

$\sec(\theta) = \frac{\sqrt{x^2+4}}{2}$

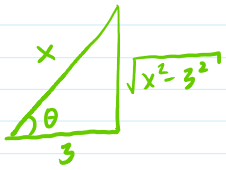
$$= \int 2^3\tan^3(\theta)\sqrt{2^2\tan^2(\theta)+2^2}\,2\sec^2(\theta)\,d\theta$$
$$= \int 8\tan^3(\theta)\sqrt{4(\tan^2(\theta)+1)}\,2\sec^2(\theta)\,d\theta$$
$$= \int 8\tan^3(\theta)\sqrt{4\sec^2(\theta)}\,2\sec^2(\theta)\,d\theta$$
$$= \int 8\tan^3(\theta)\,2\sec(\theta)\,2\sec^2\theta\,d\theta$$
$$= \int 32\sec^3(\theta)\tan^3(\theta)\,d\theta$$
$$= \int 32\sec^2(\theta)\tan^2(\theta)\sec(\theta)\tan(\theta)\,d\theta$$
$$= \int 32\sec^2(\theta)(\sec^2\theta-1)\sec(\theta)\tan(\theta)\,d\theta$$

↓ let $u = \sec(\theta)$, $du = \tan(\theta)\sec(\theta)\,d\theta$

$$= 32\int u^2(u^2-1)\,du$$
$$= 32\int (u^4-u^2)\,du$$
$$= 32\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C$$

$$= 32\int (u^4 - u^2)\,du$$
$$= 32\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C$$
$$= 32\left(\frac{\sec^5(\theta)}{5} - \frac{\sec^3(\theta)}{3}\right) + C$$
$$= 32\left(\frac{(x^2+4)^{5/2}}{5} - \frac{(x^2+4)^{3/2}}{3}\right) + C$$

- $\int \sqrt{x^2-9}\,dx = \int \sqrt{x^2-3^2}\,dx$



$\cos(\theta) = \frac{3}{x}$

$x = \frac{3}{\cos(\theta)}$

$x = 3\sec(\theta)$

$dx = 3\tan(\theta)\sec(\theta)\,d\theta$

$\sec(\theta) = \frac{x}{3}$

$\tan(\theta) = \frac{\sqrt{x^2-3^2}}{3}$

$$= \int \sqrt{3^2\sec^2(\theta) - 3^2}\; 3\tan(\theta)\sec(\theta)\,d\theta$$
$$= \int \sqrt{3^2(\sec^2(\theta)-1)}\; 3\tan(\theta)\sec(\theta)\,d\theta$$
$$= \int 3\sqrt{\sec^2(\theta)-1}\; 3\tan(\theta)\sec(\theta)\,d\theta$$
$$= \int 3\sqrt{\tan^2(\theta)}\; 3\tan(\theta)\sec(\theta)\,d\theta$$
$$= \int 9\tan(\theta)\tan(\theta)\sec(\theta)\,d\theta$$
$$= \int 9\tan^2(\theta)\sec(\theta)\,d\theta$$
$$= \int 9(\sec^2(\theta)-1)\sec(\theta)\,d\theta$$
$$= \int \left(9\sec^3(\theta) - 9\sec(\theta)\right)d\theta$$
$$= 9\int \sec^3(\theta)\,d\theta - 9\int \sec(\theta)\,d\theta$$

Use reduction formula for $\sec^3(\theta)$

$$= 9\left(\frac{1}{3-1}\sec^{3-2}(\theta)\tan(\theta) + \frac{3-2}{3-1}\int \sec^{3-2}(\theta)\,d\theta\right) - 9\int \sec(\theta)\,d\theta$$
$$= 9\left(\frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\int \sec(\theta)\,d\theta\right) - 9\int \sec(\theta)\,d\theta$$
$$= \frac{9}{2}\sec(\theta)\tan(\theta) - \frac{9}{2}\int \sec(\theta)\,d\theta$$

$\int \sec(\theta)\,d\theta$

$$= \int \sec(\theta)\left(\frac{\sec(\theta)+\tan(\theta)}{\sec(\theta)+\tan(\theta)}\right)d\theta$$
$$= \int \frac{\sec^2(\theta) + \sec(\theta)\tan(\theta)}{\sec(\theta)+\tan(\theta)}\,d\theta$$

Let $u = \sec(\theta) + \tan(\theta)$

$du = (\sec(\theta)\tan(\theta) + \sec^2(\theta))\,d\theta$

$$= \int \frac{1}{u}\,du$$
$$= \ln(|u|) + C$$
$$= \ln(|\sec(\theta)+\tan(\theta)|) + C$$

$$= \frac{9}{2}\sec(\theta)\tan(\theta) - \frac{9}{2}\ln(|\sec(\theta)+\tan(\theta)|) + C$$
$$= \frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{x^2-9}}{3}\right) - \frac{9}{2}\ln\left(\left|\frac{x}{3} + \frac{\sqrt{x^2+9}}{3}\right|\right) + C$$
$$= \frac{1}{2}x\sqrt{x^2-9} - \frac{9}{2}\ln\left(\frac{1}{3}|x + \sqrt{x^2+9}|\right) + C$$

Mini-Activities

1. $\int \sqrt{x^2+1}\,dx$

Mini – Activities

1. $\int \frac{\sqrt{x^2+1}}{x}\,dx$

2. $\int \frac{x}{\sqrt{x^4-16}}\,dx$; Hint: use u-sub first and then apply trig sub.

3. $\int \frac{\sqrt{x-1}}{x}\,dx$; Hint: use u-sub first and then apply trig sub.

4. $\int \frac{1}{\sqrt{x^2-4x}}\,dx$; Hint: complete the square of x^2-4x, and then apply trig sub.