3.4 Partial Fractions Cont.

Monday, October 24, 2022

Objectives:

1. Continue on the method of partial fractions.
2. Repeated and non-repeated linear factors.

Example from last time:

- $\displaystyle\int \frac{5x-4}{x^2-x-2}\,dx = \int\left(\frac{2}{x-2} + \frac{3}{x+1}\right)dx$

  $\downarrow$ non-repeated linear factor

  $\displaystyle = \int \frac{2}{x-2}\,dx + \int \frac{3}{x+1}\,dx$

  $= 2\ln(x-2) + 3\ln(x+1) + C$

- $\displaystyle\int \frac{5x-2}{(x+3)^2}\,dx = \int\left(\frac{5}{x+3} - \frac{17}{(x+3)^2}\right)dx$

  $\downarrow$ repeated linear factor

  $\displaystyle = \int \frac{5}{x+3}\,dx - \int \frac{17}{(x+3)^2}\,dx$

  $\displaystyle = 5\ln(x+3) + \frac{17}{x+3} + C$

Non-repeated linear factors

Example:

- $\displaystyle\int \frac{3x+2}{x^3-x^2-2x}\,dx$

Numerator: $3x+2 \rightarrow \deg(3x+2) = 1$
Denominator: $x^3-x^2-2x \rightarrow \deg(x^3-x^2-2x) = 3$

Since $\deg(3x+2) < \deg(x^3-x^2-2x)$, begin factoring the x^3-x^2-2x term, and apply partial fraction decomposition.

$$\frac{3x+2}{x^3-x^2-2x} = \frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}$$

$$\cancel{x(x-2)(x+1)}\,\frac{3x+2}{\cancel{x(x-2)(x+1)}} = \frac{A}{\cancel{x}}\,\cancel{x}(x-2)(x+1) + \frac{B}{\cancel{x-2}}\,x\cancel{(x-2)}(x+1) + \frac{C}{\cancel{x+1}}\,x(x-2)\cancel{(x+1)}$$

$$3x+2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2)$$

Now, solve for A, B, and C.

The strategy:

$$3x+2 = A(x^2-x-2) + B(x^2+x) + C(x^2-2x)$$
$$= Ax^2 - Ax - 2A + Bx^2 + Bx + Cx^2 - C2x$$
$$= Ax^2 + Bx^2 + Cx^2 - Ax + Bx - C2x - 2A$$
$$0x^2 + 3x+2 = x^2(A+B+C) + x(-A+B-2C) - 2A$$

coefficients of x^2: $0 = A+B+C$
of x: $3 = -A+B-2C$
of 1: $2 = -2A$

System of linear equations
3 equations, 3 unknowns can be solved.

$\downarrow$

start with the easy ones: $2 = -2A$
$A = -1$

$\downarrow$

solve the other ones

$0 = -1 + B + C$
$3 = -(-1) + B - 2C$

$1 = B + C \rightarrow B = 1 - C \rightarrow B = 1-(-1/3) = 4/3$
$2 = B - 2C \rightarrow 2 = (1-C) - 2C$
$2 = 1 - 3C$
$1 = -3C$
$C = -1/3$

$$A=-1, \quad B = 4/3, \quad C = -1/3$$

So,

$$\frac{3x+2}{x^3-x^2-2x} = \frac{-1}{x} + \frac{4/3}{x-2} + \frac{-1/3}{x+1}$$

$$\int \frac{3x+2}{x^3-x^2-2x}\,dx = \int\left(-\frac{1}{x} + \frac{4/3}{x-2} - \frac{1/3}{x+1}\right)dx$$

$$= \int -\frac{1}{x}\,dx + \int \frac{4/3}{x-2}\,dx - \int \frac{1/3}{x+1}\,dx$$

$$= -\ln(x) + \frac{4}{3}\ln(x-2) - \frac{1}{3}\ln(x+1)$$

<u>Repeated linear factors</u>

Example:

- $\int \frac{x-2}{(2x-1)^2(x-1)}\,dx$

Numerator: $\deg(x-2) = 1$
denominator: $\deg((2x-1)^2(x-1)) = 3$

$\deg(x-2) < \deg((2x-1)^2(x-1))$

the decomposition:

$$\frac{x-2}{(2x-1)^2(x-1)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{(x-1)}$$

$$\frac{x-2}{(2x-1)^2(x-1)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{(x-1)}$$

common denominator:

$$x-2 = A(2x-1)(x-1) + B(x-1) + C(2x-1)^2$$

equating coefficients

$$0x^2 + x - 2 = (2A+4C)x^2 + (-3A+B-4C)x + (A-B+C)$$

$$\left.\begin{aligned} 0 &= 2A+4C \\ 1 &= -3A+B-4C \\ -2 &= A-B+C \end{aligned}\right\}$$ solving this yeilds $A=2,\ B=3,\ C=-1.$

So.

$$\int \frac{x-2}{(2x-1)^2(x-1)}\,dx = \int\left(\frac{2}{2x-1} + \frac{3}{(2x-1)^2} - \frac{1}{x-1}\right)dx$$

$$= \ln(2x-1) - \frac{3}{2(2x-1)} - \ln(x-1) + C$$

Mini-Activity

Use partial fraction decomposition to evaluate the following integrals

1. $\displaystyle\int \frac{x-3}{x+2}\,dx$

2. $\displaystyle\int \frac{\cos(x)}{\sin^2(x) - \sin(x)}\,dx$ Hint: use u-sub first and then partial fractions

3. $\displaystyle\int \frac{x+1}{(x+3)(x-2)}\,dx$

4. $\displaystyle\int \frac{x+2}{(x+3)^3(x-4)^2}\,dx$