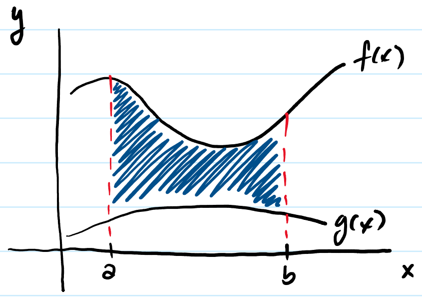
2.1 Area Between Curves

Monday, October 31, 2022

Objectives:

1. Determine the area of a region between two curves
2. Find the area of a compound region.

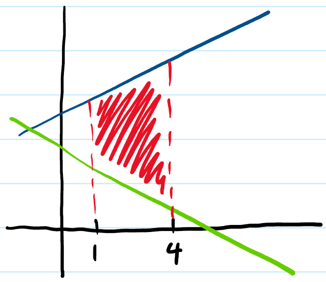
Area between two curves



$$A = \int_a^b [f(x) - g(x)]\,dx \quad \text{for } f(x) \geq g(x)$$

Example:

- Given $f(x) = x+4$ and $g(x) = 3 - x/2$ in the interval $[1,4]$.



$$A = \int_1^4 [x+4 - (3 - x/2)]\,dx$$

$$= \int_1^4 \left[\frac{3x}{2} + 1\right] dx$$

$$= \int_1^4 \frac{3x}{2}\,dx + \int_1^4 dx$$

$$= \frac{3x^2}{4}\Big|_1^4 + x\Big|_1^4$$

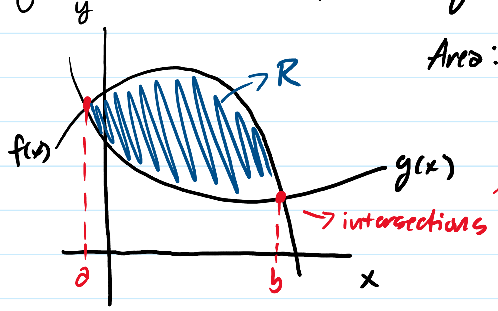
$$= \frac{3(4)^2}{4} + 4 - \left(\frac{3(1)^2}{4} + 1\right)$$

$$= 16 - \frac{7}{4}$$

$A = \frac{57}{4}$

Finding the Area of a Region between two curves.

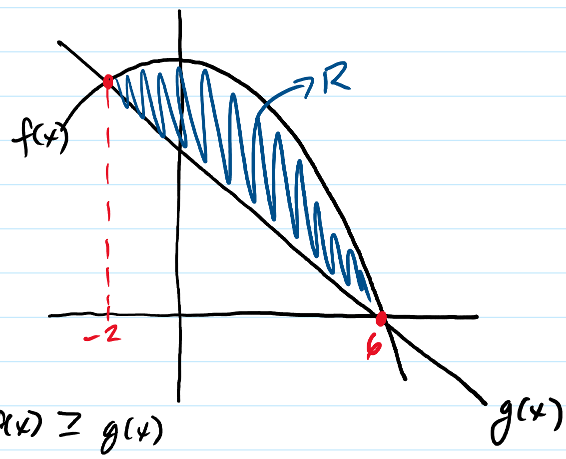
Region: $f(x) \geq g(x)$



Area: $A = \int_a^b [f(x) - g(x)]\,dx$

Example:

- Given $f(x) = 9 - (x/2)^2$ & $g(x) = 6 - x$.



Intersections:

$$f(x) = g(x)$$
$$9 - (x/2)^2 = 6 - x$$
$$9 - x^2/4 = 6 - x$$
$$36 - x^2 = 24 - 4x$$
$$x^2 - 4x - 12 = 0$$
$$(x-6)(x+2) = 0$$
$$\downarrow$$
$$x = 6 \text{ \& } x = -2$$

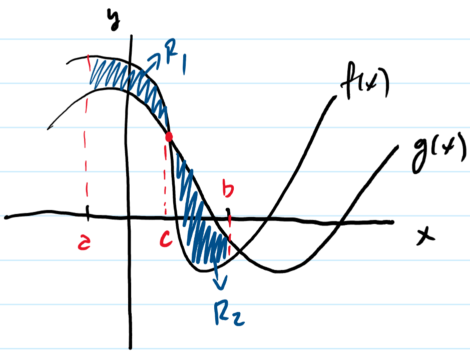
$f(x) \geq g(x)$

$$A = \int_{-2}^{6} \left[9 - (x/2)^2 - (6-x)\right] dx$$
$$= \int_{-2}^{6} \left[9 - \frac{x^2}{4} - 6 + x\right] dx$$
$$= \int_{-2}^{6} \left[3 - \frac{x^2}{4} + x\right] dx$$
$$= \int_{-2}^{6} 3\,dx - \int_{-2}^{6} \frac{x^2}{4}\,dx + \int_{-2}^{6} x\,dx$$
$$= 3x - \frac{x^3}{12} + \frac{x^2}{2} \Big|_{-2}^{6}$$

$A = \frac{64}{3}$

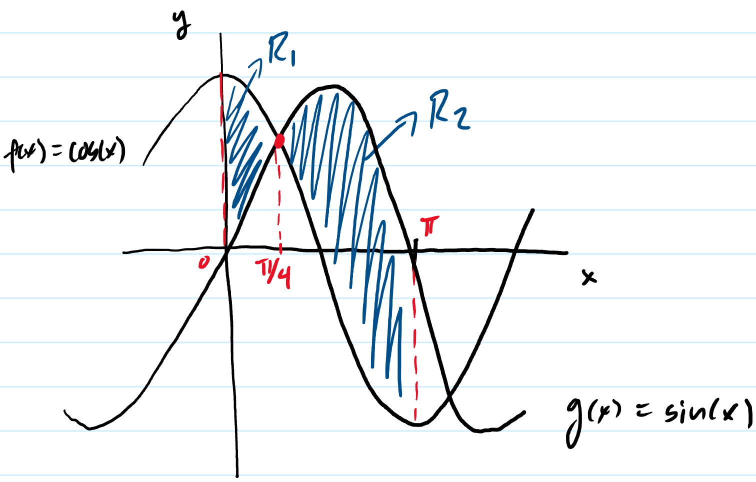
Finding the Areas of Compound Regions

Area of a region that cross



- Example:

  If R is a region between $f(x) = \sin(x)$ and $g(x) = \cos(x)$ over the interval $[0, \pi]$, find the area of R.



Intersection: $x = \pi/4$ because $\cos(\pi/4) = \sin(\pi/4)$

Two bounds: @ R_1, $x \in [0, \pi/4]$, $\cos(x) \geq \sin(x) \rightarrow |\sin(x) - \cos(x)| = \cos(x) - \sin(x)$

@ R_2, $x \in [\pi/4, \pi]$, $\sin(x) \geq \cos(x) \rightarrow |\sin(x) - \cos(x)| = \sin(x) - \cos(x)$

$$A = \int_a^b |\sin(x) - \cos(x)|\, dx$$

$$= \int_0^{\pi} |\sin(x) - \cos(x)|\, dx$$

$$= \int_0^{\pi/4} (\cos(x) - \sin(x))\, dx + \int_{\pi/4}^{\pi} (\sin(x) - \cos(x))\, dx$$

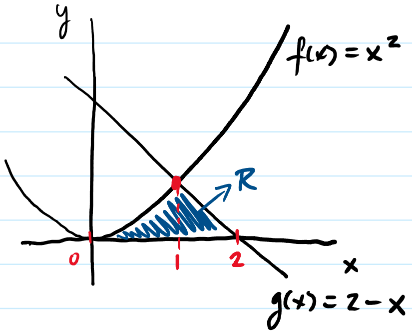
$$= \int_0^{\pi/4} (\cos(x) - \sin(x))\,dx + \int_{\pi/4}^{\pi} (\sin(x) - \cos(x))\,dx$$

$$= [\sin(x) + \cos(x)]\Big|_0^{\pi/4} + [-\cos(x) - \sin(x)]\Big|_{\pi/4}^{\pi}$$

$$A = 2\sqrt{2}$$

Finding the Area of a Complex Region

Example:



Intersection: $x^2 = 2 - x$

$$x^2 + x - 2 = 0$$

$$(x-1)(x+2) = 0$$

$$x = 1, \quad x = -2$$

@ A_1, $x = [0, 1]$, $f(x) = x^2$

@ A_2, $x = [1, 2]$, $g(x) = 2 - x$

$$A_1 = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$A_2 = \int_1^2 (2 - x)\,dx = 2x - \frac{x^2}{2}\Big|_1^2 = \frac{1}{2}$$

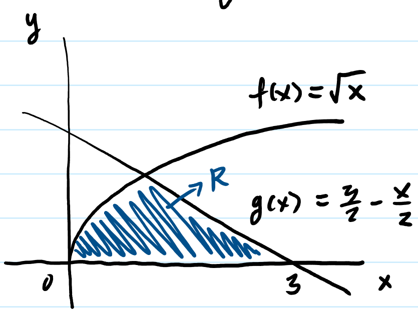
total Area: $A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

Mini-Activities

1. If R is the region bounded above by the graph of the function $f(x) = x$ and below by the graph of the function $g(x) = x^4$, find the area of region R. Draw the functions and shade R.

2. If R is the region between the graphs of the functions $f(x) = \sin(x)$ and $g(x) = \cos(x)$ over the interval $[\pi/2, 2\pi]$, find the region R. Draw the functions and shade R.

3. Consider the graph below



Find the area of R.

4. Given $f(x) = 1/x$, $g(x) = 1/x^2$, and $x = 3$

sketch the equations and shade the area of the region between the curves. Determine its area