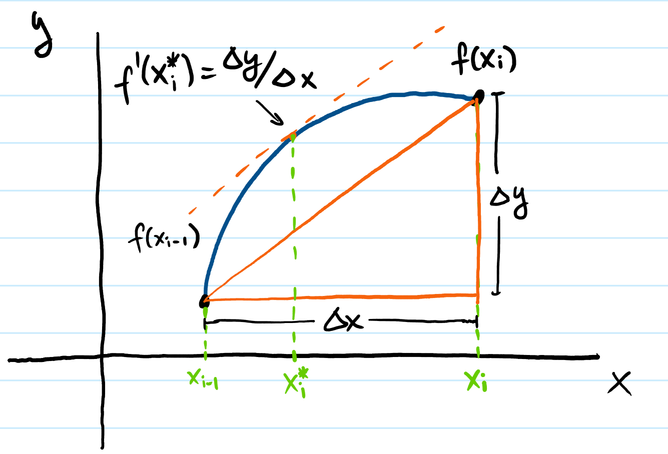
# 2.4 Arc Length of a Curve and Surface Area

Monday, November 14, 2022

Objectives:

1. Determine the length of a curve between two points
2. Find surface area of a solid of revolution.

## Arc Length of the Curve $y = f(x)$




Segment length $\sqrt{(\Delta x)^2 + (\Delta y)^2}$

$$\Delta x \sqrt{1 + ((\Delta y)^2/(\Delta x)^2)}$$

Mean Value theorem: There exist $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = \dfrac{\Delta y}{\Delta x}$.

We can rewrite the segment length as

$$\Delta x \sqrt{1 + (f'(x_i^*))^2}$$

We need to add all lengths of all line segments

$$\text{Arc length} \approx \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\, \Delta x$$

$$= \lim \sum^{n} \sqrt{1 + (f'(x_i^*))^2}\, \Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+f'(x_i^*)^2}\,\Delta x$$

$$\text{Arc length} = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

Example:

Let $f(x) = 2x^{3/2}$. Find the arc length of $f(x)$ over the interval $[0,1]$.

Find derivative: $f'(x) = 3x^{1/2}$.

$$\text{Arc length} = \int_0^1 \sqrt{1+(3x^{1/2})^2}\,dx$$

$$= \int_0^1 \sqrt{1+9x}\,dx$$ use u-sub

Let $u = 1+9x$ & $du = 9\,dx$

$$= \int_1^{10} \sqrt{u}\,\frac{1}{9}\,du$$

$$= \frac{1}{9}\int_1^{10} u^{1/2}\,du$$

$$= \frac{1}{9}\,\frac{u^{3/2}}{3/2}\Bigg|_1^{10}$$

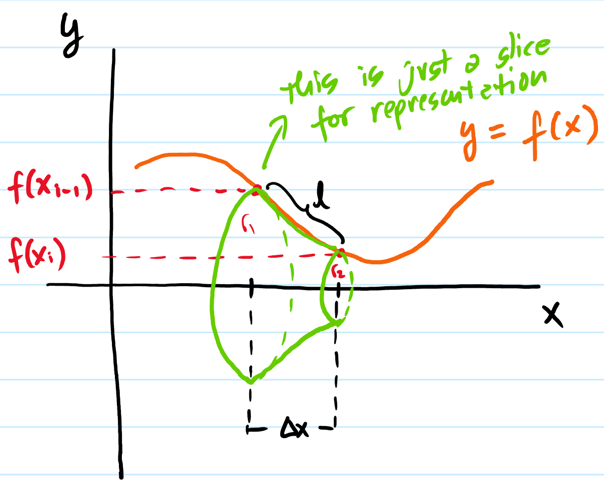
$$= \frac{2}{27}(10)^{3/2} - \frac{2}{27}(1)^{3/2}$$

$$\text{Arc length} = \frac{2}{27}\left(10^{3/2}-1\right)$$

## Area of a surface of revolution

$y$

is just a slice




$$S = \pi(r_1 + r_2)\,l$$
$$= \pi\left(f(x_{i-1}) + f(x_i)\right)\Delta x\sqrt{1 + (\Delta y/\Delta x)^2}$$
$$= \pi\left(f(x_{i-1}) + f(x_i)\right)\Delta x\sqrt{1 + (f'(x_i^*))^2}$$

↘ by mean value theorem

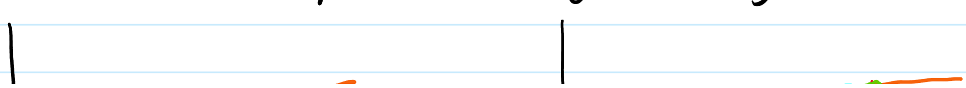
$$S = \underbrace{2\pi\left(f(x_i^{**})\right)}_{\text{by the intermediate value theorem.}}\Delta x\sqrt{1 + (f'(x_i^*))^2}$$

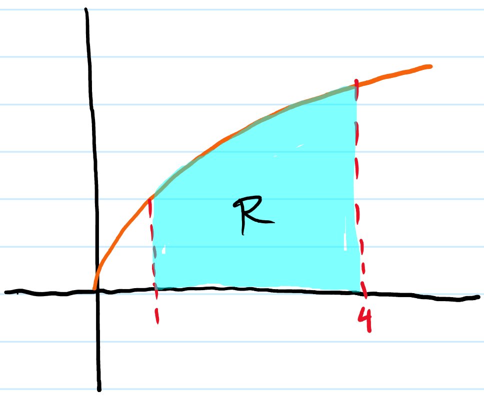
$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi\left(f(x_i^{**})\right)\Delta x\sqrt{1 + (f'(x_i^*))^2}$$

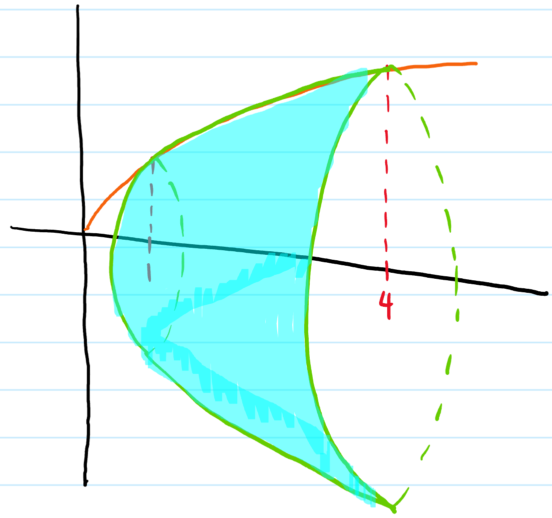
Surface Area of solid of revolution
$$= \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi\left(f(x_i^{**})\right)\Delta x\sqrt{1 + (f'(x_i^*))^2}$$
$$= \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx$$

Example:

Let $f(x) = \sqrt{x}$ over the interval $[1,4]$. Find the surface area of the surface generated by revolving the graph around the x-axis.






We have $f(x) = \sqrt{x}$. So $f'(x) = \dfrac{1}{2\sqrt{x}}$.

$$\text{Surface area} = \int_1^4 2\pi \sqrt{x} \sqrt{1 + \left(\tfrac{1}{2\sqrt{x}}\right)^2}\, dx$$

$$= \int_1^4 2\pi \sqrt{x} \sqrt{1 + \tfrac{1}{4x}}\, dx$$

$$= \int_1^4 2\pi \sqrt{x + \frac{1}{4}}\, dx$$

let $u = x + 1/4$

$du = dx$

bounds: $x = 1 : u = 1 + 1/4 = 5/4$

$x = 4 : 4 + 1/4 = 17/4$

$$= \int_{5/4}^{17/4} 2\pi \sqrt{u}\, du$$

$$= 2\pi \frac{2}{3} u^{3/2} \Big|_{5/4}^{17/4}$$

$$\text{Surface Area} = \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$$

## Mini-Activity

1. Let $f(x) = \sin(x)$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, \pi]$. Set-up only the integral.
2. Let $g(y) = 1/y$. Calculate the arc length of the graph of $g(y)$ over the interval $[1, 4]$. Set-up only the integral.

graph of $f(x)$ over the interval [illegible]. Set-up only the integral.

2. Let $g(y) = 1/y$. Calculate the arc length of the graph of $g(y)$ over the interval $[1,4]$. Set-up only the integral.
3. Let $f(x) = \sqrt{1-x}$ over the interval $[0, 1/2]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis.
4. Let $g(y) = \sqrt{9-y^2}$ over the interval $y \in [0,2]$. Find the surface generated by revolving the graph of $g(y)$ around the y-axis.