6.1 Power Series

Wednesday, November 16, 2022

Objectives:

1. Identify a power series and give examples.
2. Determine the radius of convergence and interval of convergence.

Form of a Power Series

A series of the form

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \cdots$$

is a power series centered at $x=0$.

A series of the form

$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots$$

is a power series centered at $x=a$.

Power Series Examples

- $$\sum_{n=0}^{\infty} x^n = \underset{n=0}{1} + \underset{n=1}{x} + \underset{n=2}{x^2} + \underset{n=3}{x^3} + \cdots$$

  ↳ x is a variable

  ↳ this one is a geometric series with ratio $r=x$.

  ↳ we know that if $|x|<1$, it converges.
  if $|x| \geq 1$, it diverges.

  ↳ centered at $x=0$.

- $$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \underset{n=0}{1} + \underset{n=1}{x} + \underset{n=2}{\frac{x^2}{2!}} + \underset{n=3}{\frac{x^3}{3!}} + \cdots$$

  ↳ centered at $x=0$.

- $$\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n} = \underset{n=0}{1} + \underset{n=1}{\frac{x-2}{2\cdot 3}} + \underset{n=2}{\frac{(x-2)^2}{3\cdot 3^2}} + \underset{n=3}{\frac{(x-2)^3}{4\cdot 3^3}} + \cdots$$

↳ centered at $x=2$.

Convergence of a Power Series

Consider the power series $\sum_{n=0}^{\infty} C_n(x-a)^n$.

The series satisfies exactly one of the following properties:

i. The series converges at $x=a$ and diverges for all $x \neq a$.
ii. The series converges for all real numbers x.
iii. There exists a real number $R>0$ such that the series converges if $|x-a|<R$ and diverges if $|x-a|>R$. At values x where $|x-a|=R$, the series may converge or diverge. R is called the radius of convergence roc.

Definitions and terminologies

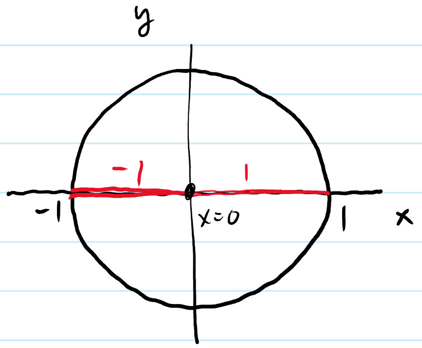
1. If a power series converges only for $x=a$, the roc is defined to be $R=0$.
2. If the power series converges for all x, then the roc is $R=\infty$.
3. If the power series converges for values of x which $|x-a|<R$ or $a-R<x<a+R$, the roc is R.
4. The interval of convergence is the interval $(a-R, a+R)$ including the endpoint where the power series converges.

Example:

$$\sum_{n=0}^{\infty} x^n \longrightarrow \text{geometric series}$$

↳ centered at $x=0$.

roc: $R=1$ and ioc: $(0-1, 0+1) = (-1, +1)$.



Given $-1 < x < 1$

$$R = \frac{1-(-1)}{2} = 1$$

Method for computing radius of Convergence

Use the Ratio Test.

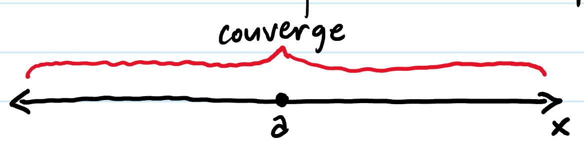
Step 1: Let $a_n = C_n(x-a)^n$
and $a_{n+1} = C_{n+1}(x-a)^{n+1}$

Step 2: Simplify ratio $\dfrac{|a_{n+1}|}{|a_n|} = \dfrac{|C_{n+1}(x-a)^{n+1}|}{|C_n(x-a)^n|} = \dfrac{C_{n+1}}{C_n}(x-a)$

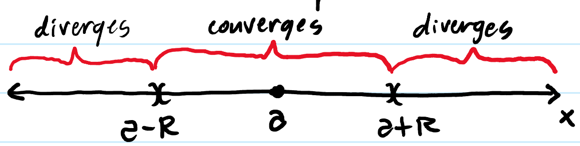
Step 3: Compute $\rho = \lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|}$

Step 4: Interpret the results.

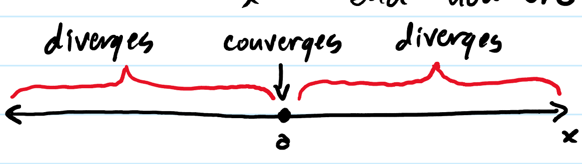
i. If $\rho = 0$, then $R = \infty$, the power series converges for all values of x.



ii. if $\rho = N \cdot |x-a|$, where N is a finite, positive number, then $R = \dfrac{1}{N}$. The interval of convergence includes $(a - \frac{1}{N}, a + \frac{1}{N})$ and possibly the end points.



iii. If $\rho \to \infty$, then the $R = 0$. The power series converges at $x = a$ and nowhere else.



About the End points:

If you have interval of convergence $(a - \frac{1}{N}, a + \frac{1}{N})$, the end points $x = a - \frac{1}{N}$ and $x = a + \frac{1}{N}$ may or may not converge.
To determine whether the endpoints converge, plug them to the power series one at a time and use a convergence test.

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Examples:

- $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ $\longrightarrow$ power series of the form $c_n x^n$ where $c_n = \frac{1}{n!}$.

$a_n = \frac{x^n}{n!}$, $a_{n+1} = \frac{x^{n+1}}{(n+1)!}$

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

$(n+1)! = (n+1)n!$

$$= \lim_{n\to\infty} \frac{|x^{n+1}/(n+1)!|}{|x^n/n!|}$$

$$= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{n+1} \right|$$

$$= |x| \lim_{n\to\infty} \frac{1}{n+1}$$

$\rho = 0 < 1$ for all values of x.

Therefore, the series converges for all x with $R = \infty$.
Interval of convergence is $(-\infty, \infty)$.

- $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}$

$a_n = \frac{(x-2)^n}{(n+1)3^n}$, $a_{n+1} = \frac{(x-2)^{n+1}}{(n+2)3^{n+1}}$

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+2)3^{n+1}}}{\frac{(x-2)^n}{(n+1)3^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{(x-2)^n} \right|$$

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$$= \lim_{n\to\infty} \left| \frac{(x-2)(n+1)}{3(n+2)} \right|$$

$$= |x-2| \cancel{\lim_{n\to\infty} \frac{n+1}{3(n+2)}} \ \frac{1}{3}$$

$$\rho = \frac{|x-2|}{3} \quad , \quad N = \frac{1}{3}$$

1. $\rho < 1$ if $|x-2| < 3$.

Since $|x-2| < 3$, then $-3 < x-2 < 3$
$-1 < x < 5$.

the series converges if $-1 < x < 5$.

2. $\rho > 1$ if $|x-2| > 3$, the series diverges if $x < -1$, $x > 5$.
3. inconclusive if $\rho = 1$, that is when $x = -1$ or $x = 5$.

Therefore, the radius of convergence is $R = 1/N = 1/\tfrac{1}{3} = 3$.

End points: @ $x = -1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$ $\rightarrow$ converges by the alternating series test.

So the series conv. at $x = -1$.

@ $x = 5$: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1}$ $\rightarrow$ diverges because it's a harmonic series.

So the series diverges at $x = 5$.

Finally, the interval of convergence is $[-1, 5)$.

Mini-Activities

Find the radius of convergence and the interval of convergence.

1. $\displaystyle\sum^{\infty} n!\, x^n$

1. $\sum_{n=1} n!\,x^n$

2. $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

3. $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$

4. $\sum_{n=1}^{\infty} \frac{nx^n}{e^n}$