6.3 Taylor and Maclaurin Series

Monday, November 21, 2022

Objectives:

1. Continue on the Properties of power series
2. Introduction to Taylor Series.
3. Finding a Taylor polynomial of a given order for a function.

Properties of Power Series.

3. Differentiating and Integrating power series

Suppose $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges on the interval $(a-R, a+R)$ for some $R > 0$.

Let $$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$
$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$
for $|x-a| < R$.

derivative $$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}$$
$$= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$$

integral $$\int f(x) = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

$$= C + c_0(x-a) + \frac{c_1(x-a)^2}{2} + \dots$$

Example:

- $$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1.$$

2. Find the power series of $g(x) = \frac{1}{(1-x)^2}$ on $(-1, 1)$.

$$f'(x) = \frac{1}{(1-x)^2} = g(x)$$

$$g(x) = \frac{1}{(1-x)^2}$$

$$= \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$= \sum_{n=0}^{\infty} \frac{d}{dx}(x^n)$$

$$= \frac{d}{dx}\left(1 + x + x^2 + x^3 + \cdots\right)$$

$$= 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$g(x) = \sum_{n=0}^{\infty} (n+1)x^n$$

b. Use part a to evaluate $\sum_{n=0}^{\infty} \frac{n+1}{4^n}$.

$$\sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}$$

So, $$\sum_{n=0}^{\infty} \frac{(n+1)}{4^n} = \sum_{n=0}^{\infty} (n+1)\left(\frac{1}{4}\right)^n$$

$$= \frac{1}{(1-1/4)^2}$$

$$= 1/(3/4)^2$$
$$= 16/9$$

- $f(x) = \ln(1+x)$, Find the power series of f by integrating the power series of f'.

$$f'(x) = \frac{1}{1+x}$$

So, $$\frac{1}{1+x} = \frac{1}{1-(-x)}$$

$$= \sum_{n=0}^{\infty} (-x)^n$$

$$= 1 - x + x^2 - x^3 + \cdots \text{ for } |x| < 1.$$

$$= 1 - x + x^2 - x^3 + \dots \quad \text{for } |x| < 1.$$

$$\int f'(x)\,dx = \int \sum_{n=0}^{\infty} (-x)^n \,dx$$
$$= \int (1 - x + x^2 - x^3 + \dots)\,dx$$
$$= C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Solve for C: Since $\ln(1+0) = 0$, then $C = 0$.

Thus, $f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$
$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

Mini-Activity

1. Differentiate the series $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$ term-by-term to find the power series for $\frac{2}{(1-x)^3}$ on $(-1,1)$.
2. Integrate the power series $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ term-by-term to evaluate $\int \ln(1+x)\,dx$.

Introduction to Taylor Series

Consider the power series

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + \dots$$

What are the coeffs?

$$f(a) = \sum_{n=0}^{\infty} C_n (a-a)^n = C_0 \longrightarrow \text{first coeff.}$$

Differentiate term-by-term

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} C_n (x-a)^n\right) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \dots$$

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$$

$$f'(a) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (a-a)^n\right) = c_1 \longrightarrow \text{2nd coeff.}$$

$$f''(x) = \frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \dots$$

$$f''(a) = 2c_2$$

$$f'''(x) = \frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + \cdots$$

$$f'''(a) = 3\cdot 2c_3.$$

So,

$$c_0 = f(a)$$
$$c_1 = f'(a)$$
$$c_2 = \frac{f''(a)}{2}$$
$$c_3 = \frac{f'''(a)}{3\cdot 2}$$

In, general $c_n = \dfrac{f^{(n)}(a)}{n!}$.

<u>Taylor series</u> for the function f is

$$\sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(a)}{n!}}_{c_n}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

The Taylor series for f at 0 is called the <u>Maclaurin series</u> for f.