6.4 Working with Taylor Series

Wednesday, November 23, 2022

Objectives

1. Finding Taylor polynomial of a given order for a function
2. Finding Taylor series of a function.
3. Recognize common Taylor series expansions.

Previously ...

Taylor Series

If f has derivatives of all orders at $x=a$, then the taylor series for the function f at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The Taylor series for f at 0 is known as the maclaurin series.

Taylor Polynomials

The nth partial sum of the Taylor series for a function f at a is known as the nth Taylor polynomial.

$$p_0(x) = f(a) \longrightarrow \text{0th partial sum}$$
$$p_1(x) = f(a) + f'(a)(x-a) \longrightarrow \text{1st partial sum}$$
$$p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \longrightarrow \text{2nd partial sum}$$
$$p_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \longrightarrow \text{3rd partial sum}$$

In general, If f has n derivatives at $x=a$, then the nth Taylor polynomial for f at a is

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The nth Taylor polynomial for f at 0 is the nth Maclaurin polynomial.

Example:

- Find the Taylor polynomials P_0, P_1, P_2, and P_3 for $f(x) = \ln(x)$ at $x=1$.

$$
\begin{aligned}
f(x) &= \ln(x) & f(1) &= 0 \\
f'(x) &= 1/x & f'(1) &= 1 \\
f''(x) &= -1/x^2 & f''(1) &= -1 \\
f'''(x) &= 2/x^3 & f'''(1) &= 2
\end{aligned}
$$

So,

$$P_0(x) = f(1) = 0$$

$$P_1(x) = f(1) + f'(1)(x-1) = x-1$$

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = (x-1) - \frac{1}{2}(x-1)^2$$

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3.$$

$\vdots$ If you keep going, you can find a pattern and get an nth Taylor polynomial in sigma notation

$$P_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{}(x-1)^k$$

Mini-Activity Part 1

1. Find the Taylor polynomials P_0, P_1, P_2 and P_3 for $f(x) = \frac{1}{x}$ at $x=1$.

2. Find the Maclaurin polynomials P_0, P_1, P_2, P_3 for the following functions and find the nth Maclaurin polynomial and write it using sigma notation.

   a. $f(x) = e^x$
   b. $f(x) = \sin(x)$

Taylor Series Expansions of common functions

- $f(x) = \frac{1}{} = \sum_{1}^{\infty} x^n$, $-1 < x < 1$

- $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad , \quad -1 < x < 1$

- $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad , \quad -\infty < x < \infty$

- $f(x) = \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad , \quad -\infty < x < \infty$

- $f(x) = \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad , \quad -\infty < x < \infty$

- $f(x) = \ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad , \quad -1 < x \leq 1$

- $f(x) = \tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad , \quad -1 \leq x \leq 1$

Deriving Maclaurin Series from Known Series

Find the maclaurin series of the function

$$f(x) = \cos(\sqrt{x})$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$$

$$= 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} + \cdots$$

Mini-Activity Part 2

3. Find the Maclaurin series of the following functions.

   a. $f(x) = \sinh(x)$ $\quad$ (Hint: $\sinh(x) = \frac{e^x - e^{-x}}{2}$)

   b. $f(x) = \sin(x^2)$