7.2 Calculus of Parametric Curves

Wednesday, November 30, 2022

Objectives:

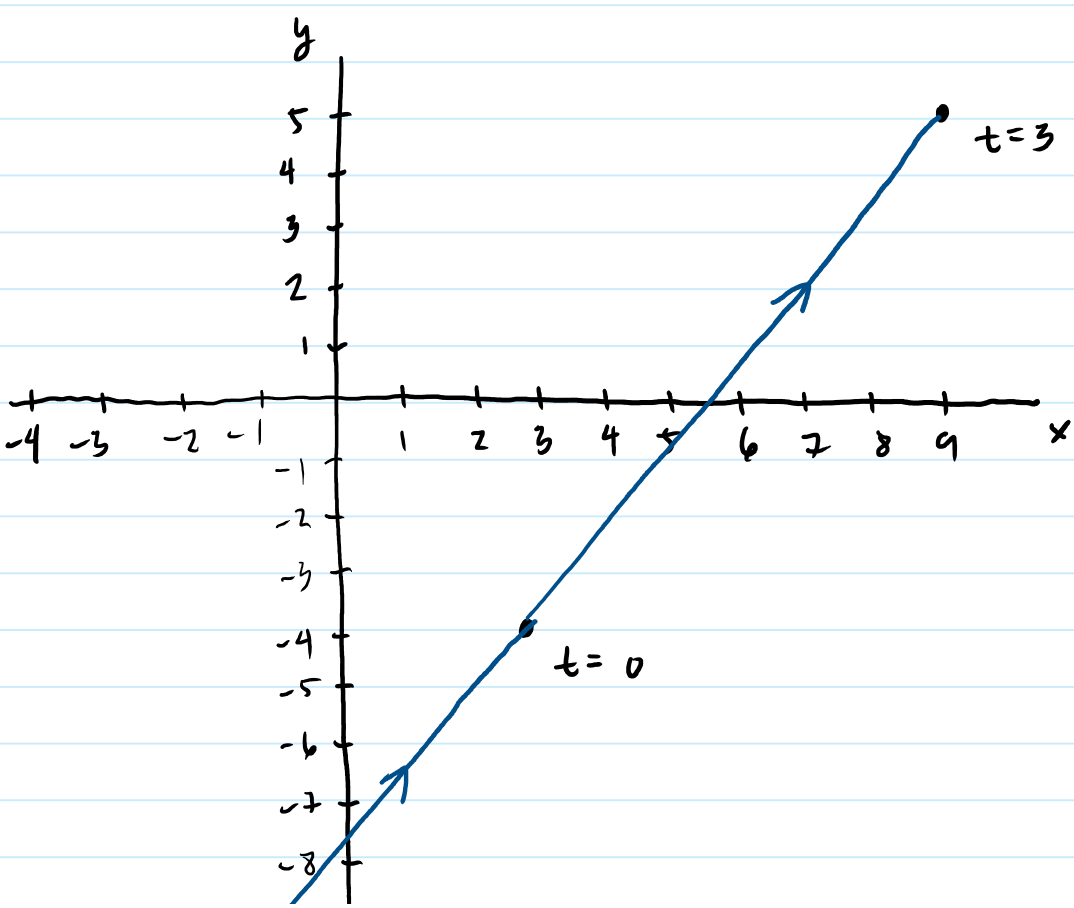
1. Determine derivatives and equations of tangents for parametric curves.

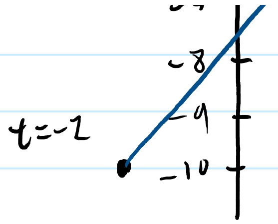
Previously...

Parametric equations

Example: $x(t) = 2t+3$, $y(t) = 3t-4$, $-2 \le t \le 3$

t	$x(t)$	$y(t)$
-2	-10	-1
0	-4	3
3	9	5





Convert $x=(t)$, $y(t)$ to $y=f(x)$

$$\left.\begin{aligned} x(t) &= 2t+3 \\ x-3 &= 2t \\ t &= \frac{x-3}{2} \end{aligned}\right\}$$ solve for t using $x(t)$.

$$\left.\begin{aligned} y(t) &= 3t-4 \\ y &= 3\left(\frac{x-3}{2}\right)-4 \\ y &= \frac{3x}{2}-\frac{9}{2}-4 \\ y &= \frac{3x}{2}-\frac{17}{2} \end{aligned}\right\}$$ solve for y using t.

the slope of y is $\frac{dy}{dx} = \frac{3}{2} \rightarrow$ using $y=f(x)$

the slope using $x=x(t)$ and $y=y(t)$ is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$\frac{dy}{dx} = \frac{3}{2}$$

<u>Derivatives of Parametric Equations</u>

Consider a plane curve defined by the parametric equations $x=x(t)$ and $y=y(t)$. Suppose that

Consider a plane curve defined by the parametric equations $x = x(t)$ and $y = y(t)$. Suppose that $x'(t)$ and $y'(t)$ exist, and assume $x'(t) \neq 0$. Then the derivative dy/dx is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}.$$

Examples:

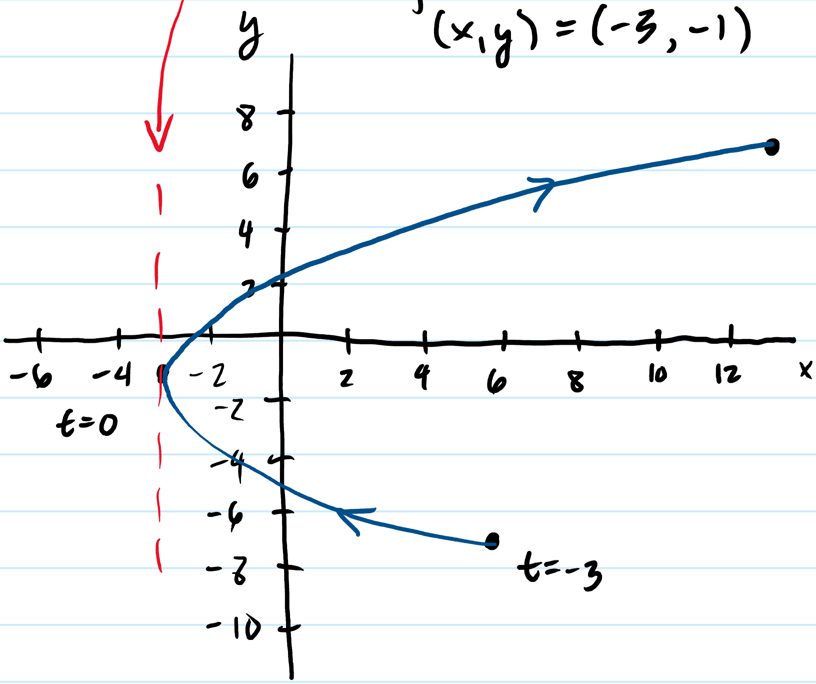
- $x(t) = t^2 - 3$, $y(t) = 2t - 1$, $-3 \leq t \leq 4$

$$dx/dt = 2t$$
$$dy/dt = 2$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2}{2t} = \frac{1}{t}$$

undefined $t=0$

$(x, y) = (-3, -1)$



- $x(t) = 2t + 1$, $y(t) = t^3 - 3t + 4$, $-2 \leq t \leq 5$

$$dx/dt = 2$$
$$dy/dt = 3t^2 - 3$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2} = \frac{3t^2}{2} - \frac{3}{2}$$

$\downarrow$

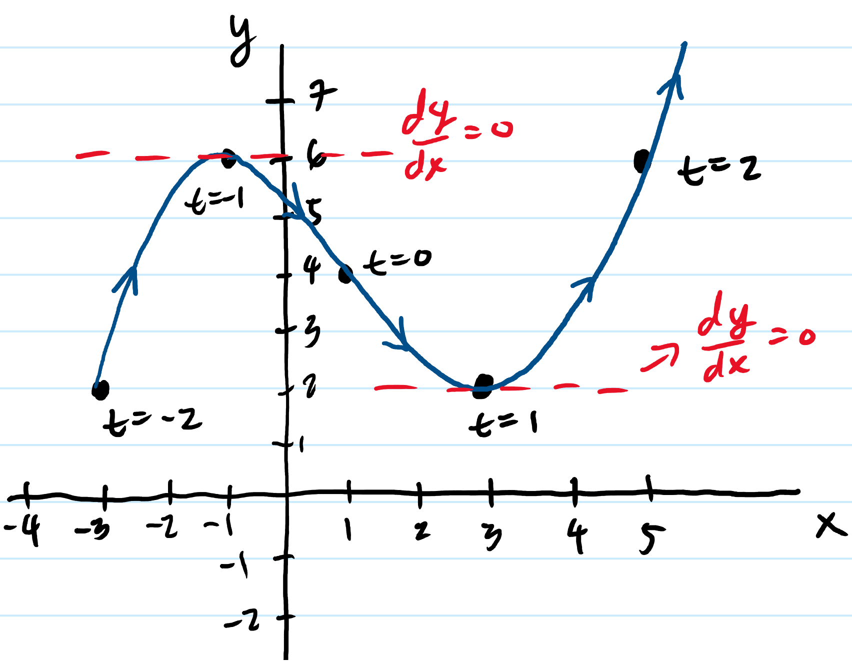
the derivative is zero when

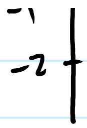
$$\frac{3t^2}{2} - \frac{3}{2} = 0$$

$$\frac{3}{2}t^2 = \frac{3}{2}$$

$$t^2 = 1 \rightarrow t = \pm 1$$

t	$x(t)$	$y(t)$
-2	-3	2
-1	-1	6
0	1	4
1	3	2
2	5	6
5	11	114





Mini-Activity Part 1

Calculate the derivative dy/dx of the following parametric equations and located any critical points.

1. $x(t) = 5\cos(t)$, $y(t) = 5\sin(t)$, $0 \le t \le 2\pi$

2. $x(t) = t^2 - 4t$, $y(t) = 2t^3 - 6t$, $-2 \le t \le 3$

Finding a Tangent Line

Example:

- $x(t) = t^2 - 3$, $y(t) = 2t - 1$, $-3 \le t \le 4$ when $t = 2$.

$$dx/dt = 2t$$
$$dy/dt = 2$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{t}$$

when $t = 2$, $\frac{dy}{dx} = \frac{1}{2}$ → this the slope of the tangent line.

So, at $t = 2$, $(x, y) = (1, 3)$.

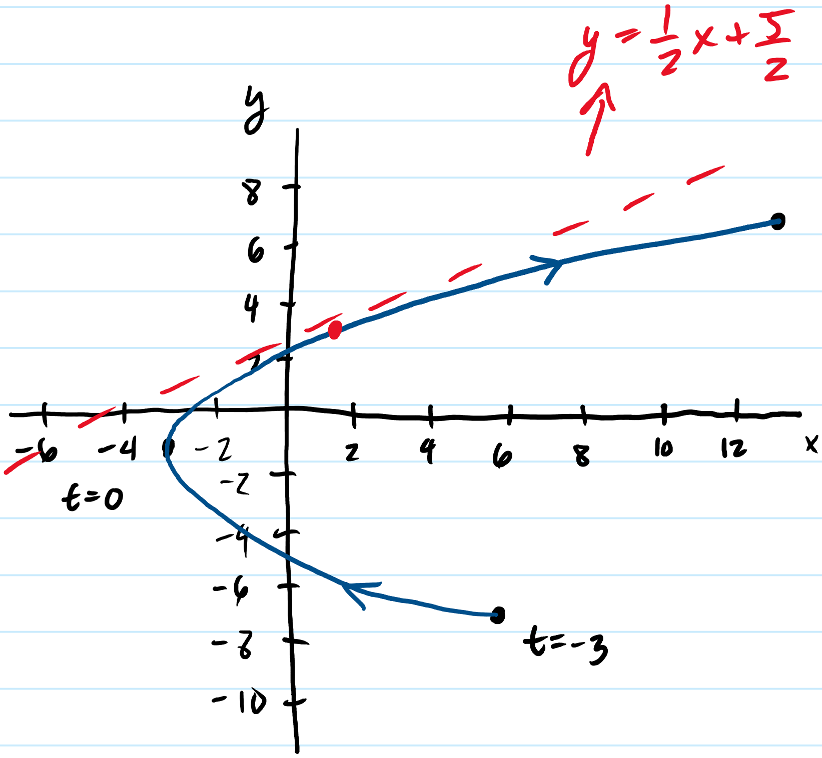
then, Apply point-slope form to find tangent line.

$$y - y_0 = \overset{\to\, dy/dx}{m}(x - x_0)$$

$$y - 3 = \tfrac{1}{2}(x - 1)$$

$$y - 3 = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x + \frac{5}{2}$$



Mini-Activity Part 2

Find the equation of the tangent line to the curve defined by the equations

$x(t) = t^2 - 4t$, $y(t) = 2t^3 - 6t$, $-2 \le t \le 10$ when $t = 5$.

Plot the parametric curve and the tangent line using Desmos.