7.2 Calculus of Parametric Curves Cont.

Friday, December 2, 2022

Objectives:

1. Find area under a parametric curve.
2. Finding the arc length of a parametric curve.
3. Finding the surface area generated by a parametric curve.

Integrals Involving parametric curves

I: Area under a Parametric Curve:

Consider the non-self-intersecting plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad a \le t \le b$$

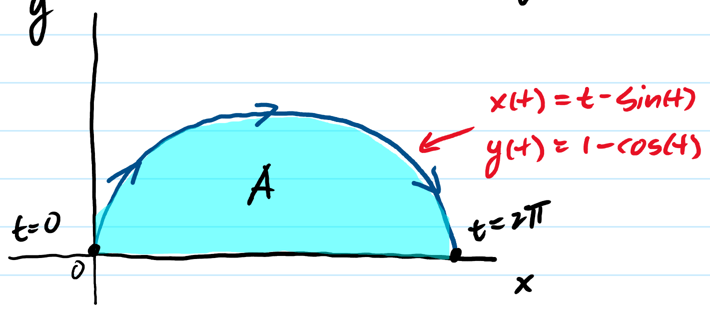
and assume that $x(t)$ is differentiable. The area under this curve is given by

$$A = \int_a^b y(t)\, x'(t)\, dt.$$

Example:

- $x(t) = t - \sin(t), \quad y(t) = 1 - \cos(t), \quad 0 \le t \le 2\pi$

this is an example of a cycloid.



$$A = \int_a^b y(t)\, x'(t)\, dt$$

$$= \int_0^{2\pi} (1-\cos(t))(1-\cos(t))\, dt$$

$$= \int_0^{2\pi} \left(1 - 2\cos(t) + \cos^2(t)\right) dt$$

$$= \int_0^{2\pi} \left(1 - 2\cos(t) + \frac{1}{2} + \frac{\cos(2t)}{2}\right) dt$$

$$= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos(t) + \frac{\cos(2t)}{2}\right) dt$$

$$= \frac{3}{2}t - 2\sin(t) + \frac{\sin(2t)}{4} \Big|_0^{2\pi}$$

$A = 3\pi$.

Mini-Activity Part 1

Find the area under the curve of the hypocycloid defined by the equations

$$x(t) = 3\cos(t) + \cos(3t), \quad y(t) = 3\sin(t) - \sin(3t), \quad 0 \le t \le \pi$$

II: Arc length of a Parametric Curve

Consider the plane curve defined by the parametric equations

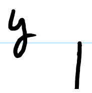
$$x = x(t), \quad y = y(t), \quad t_1 \le t \le t_2$$

and assume that $x(t)$ and $y(t)$ are differentiable functions of t. Then the arc length of this curve is given by

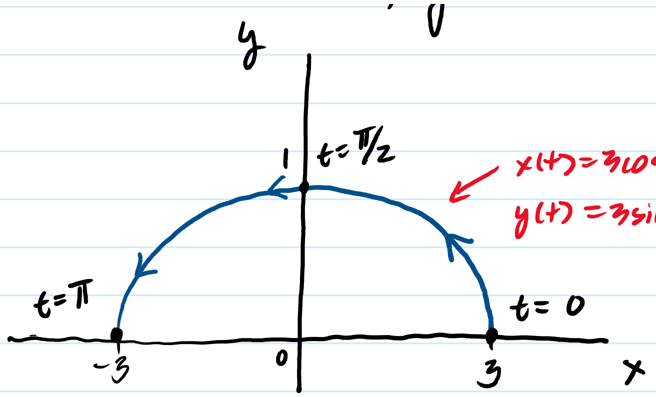
$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example:

- $x(t) = 3\cos(t), \quad y(t) = 3\sin(t), \quad 0 \le t \le \pi.$



$dx = -3\sin(t)$



$$\frac{dx}{dt} = -3\sin(t)$$

$$\frac{dy}{dt} = 3\cos(t)$$

Arc length

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{(-3\sin(t))^2 - (3\cos(t))^2}\, dt$$

$$= \int_0^{\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$$

$$= \int_0^{\pi} \sqrt{9\underbrace{(\sin^2 + \cos^2 t)}_{1}}\, dt$$

$$= \int_0^{\pi} 3\, dt$$

$$= 3t\Big|_0^{\pi}$$

$$s = 3\pi$$

Mini-Activity Part 2

Find the arc length of the curve defined by the equations

$$x(t) = 3t^2, \quad y(t) = 2t^3, \quad 1 \le t \le 3$$

III: Surface Area generated by a Parametric Curve

Recall the formula:

Recall the formula:

$$S = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$$

$\downarrow$

surface area generated by a function $y=f(x)$ from $x=a$ to $x=b$.

Now, consider the parametric equations

$$x = x(t),\ y = y(t),\quad a \le t \le b.$$

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

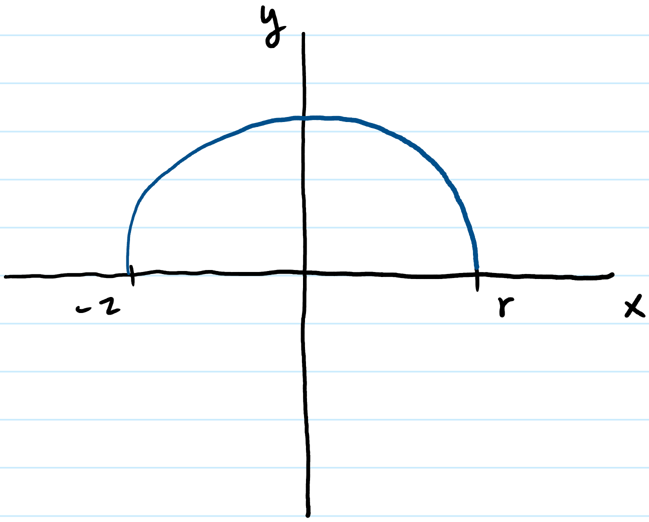
$\downarrow$

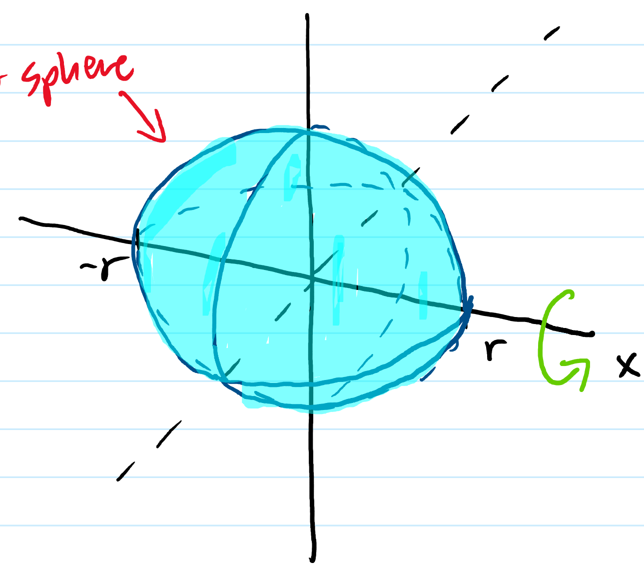
surface area generated by a parametric curve.

Example:

- $x(t) = r\cos(t)$, $y(t) = r\sin(t)$, $0 \le t \le \pi$

this is the parametric equations for a circle of radius R.
Since domain is $0 \le t \le \pi$, then its is a semi-circle.





Surface area:

$$x'(t) = -r\sin(t) \quad , \quad y'(t) = r\cos(t)$$

$$S = 2\pi \int_a^b y(t) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= 2\pi \int_0^{\pi} r\sin(t) \sqrt{(-r\sin(t))^2 + (r\cos(t))^2}\, dt$$

$$= 2\pi \int_0^{\pi} r\sin(t) \sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt$$

$$= 2\pi \int_0^{\pi} r^2 \sin(t)\, dt$$

$$= 2\pi r^2 (-\cos(t)) \Big|_0^{\pi}$$

$$= 2\pi r^2 \left(-\cancelto{-1}{\cos(\pi)} + \cancelto{1}{\cos(0)}\right)$$

$$S = 4\pi r^2 \rightarrow \text{formula for circle surface area.}$$

Mini-Activity Part 3

Find the surface area generated when the plane curve defined by the equations

$$x(t) = t^3, \quad y(t) = t^2, \quad 0 \leq t \leq 1$$

is revolved around the x-axis.