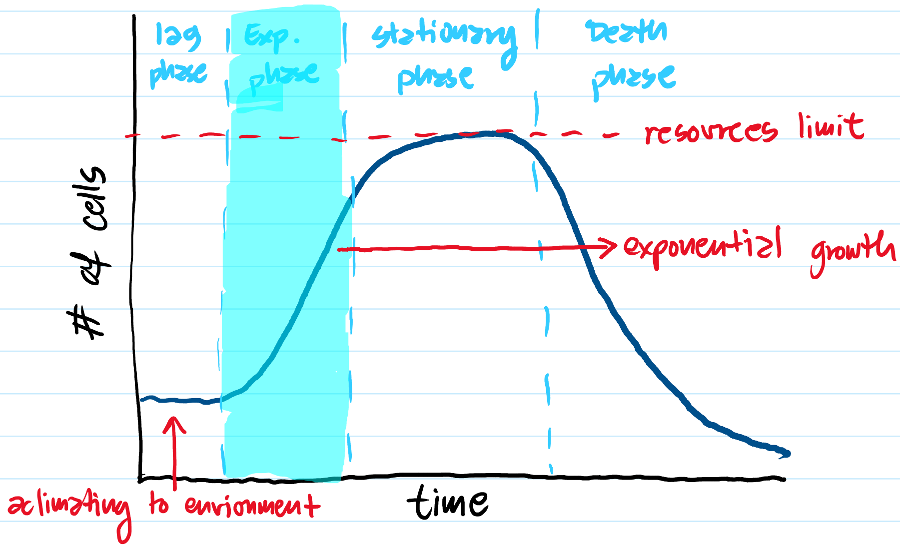
1st-Order ODEs & Equilibriums

Objectives:

1. Explain the physical meaning of 1st-order ODEs.
2. Define equilibrium solutions.

Motivating Example: Exponential growth/decay of bacteria

* Bacterial species on an environment
  → The idea is to model the growth/decay of bacteria.
  → One species
  → Repliate through the process of binary fission.



* The ODE model
  - $\frac{dP}{dt} = KP$, where $P(t)$ is the population (# of cells) at time t (minutes),
    K is the growth/decay constant, and
    with constraint $P(t) > 0$.
    → 1st-order, linear, homo, auto
  - Assumptions: (1) the species has existed for some time; meaning $P(0) > 0$,
    (2) The habitat has unlimited resources, and
    (3) the species reproduces continuously.
  - K is the parameter.
    → if $K > 0$, then you have growth
    → if $K < 0$, then you have decay
    → if $K = 0$, then no growth/decay

* Example 1: $K = 0.10$; $\frac{dP}{dt} = 0.10P$ with condition $P(0) = 10$. → exponential growth
  - When does the bacteria double in population?

    Way 1: → Limit definition of the derivative: $\frac{dP}{dt} = \lim_{h \to 0} \frac{P(t+h) - P(t)}{h}$

$\longrightarrow \frac{dP}{dt} = 0.10P$, $P(0) = 10$ and let $h = \Delta t$

$$\frac{P(t+\Delta t) - P(t)}{\Delta t} = 0.10P(t)$$

$$P_{t+\Delta t} - P_t = 0.10 P_t \Delta t$$

$$P_{t+\Delta t} = 0.10 P_t \Delta t + P_t$$

$$P_{t+\Delta t} = P_t(0.10\Delta t + 1), \quad P_0 = 10$$

Define $\Delta t = 1$ (for convenience). This means our result is an approximation.

$$P_0 = 10$$

$$P_1 = P_0(0.10(1)+1) = 10(1.1) = 11$$

$$P_2 = P_1(0.10(1)+1) = 11(1.1) = 12.1$$

$$P_3 = P_2(0.10(1)+1) = 12.1(1.1) = 13.31$$

$$\vdots$$

$$P_6 = P_5(0.10(1)+1) = 16.1051(1.1) = 17.71561$$

$$P_7 = P_6(0.10(1)+1) = 17.71561(1.1) = 19.487171 \longrightarrow t_{double} \approx 7 \text{ min.}$$

Way 2: $\longrightarrow$ General solution: $P(t) = Ce^{0.10t}$

$\longrightarrow$ specific solution: $P(0) = 10 \longrightarrow P(0) = C\cancel{e^{0.10(0)}}^{1}$

$$10 = C$$

$$P(t) = 10e^{0.10t}$$

$\longrightarrow$ Double in population since $t=0$: $P(t) = 2P(0)$

$$\cancel{10}e^{0.10t} = 2(\cancel{10})$$

$$e^{0.10t} = 2$$

$$\ln(e^{0.10t}) = \ln(2)$$

$$0.10t = \ln(2)$$

$$t_{double} = \frac{\ln(2)}{0.10} \approx 6.9315 \text{ min.}$$

- What if $P(0) = 0$? Zero cells at $t=0$.

$\longrightarrow$ Specific Solution: $\underbrace{P(t) = 0}$. population stays zero.

this is also an equilibrium solution.

Equilibrium Solutions of Autonomous 1st-order ODEs

* An equilibrium is when the solution is unchanging, meaning $\frac{dy}{dt} = 0$.

* Definition: Let $\frac{dy}{dt} = f(y)$ be an autonomous ODE

of some continuous function $y(t)$ and $f(y)$ is a function that describes the change.

Suppose y^* is a critical point. Then, 1. $f(y^*) = 0$ and

2. $y(t) = y^*$ is an equilibrium solution.

* Example 2: $\frac{dP}{dt} = \boxed{0.10P} \rightarrow f(P) = 0.10P$

- What is the equilibrium solution? Set $\frac{dP}{dt} = 0$.

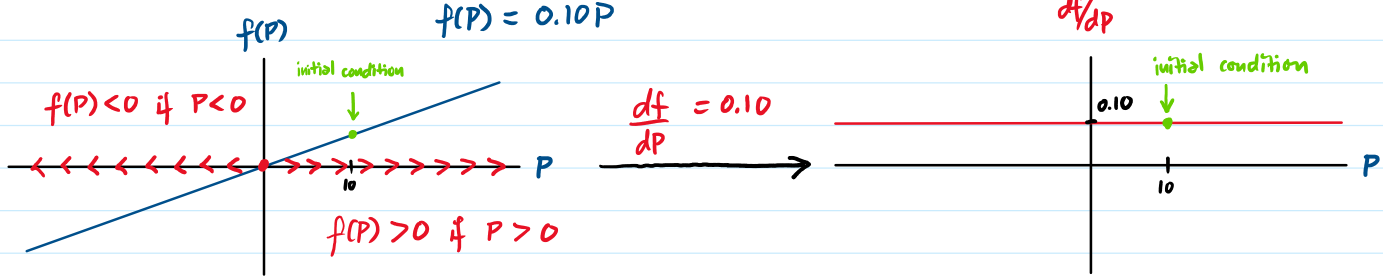
  $\longrightarrow$ $P = 0$ is a critical point because $f(0) = 0$.

  $\longrightarrow$ $P(t) = 0$ is a solution because $\frac{dP}{dt} = 0 \longrightarrow 0 = 0.10(0)$
  $0 = 0$ ✓

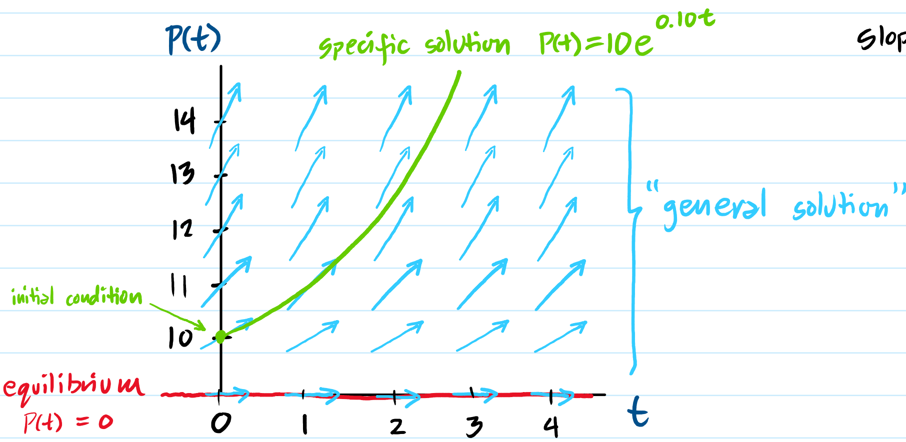
Visualizing Equilibriums

* Phase Diagram of $\frac{dP}{dt} = \boxed{0.10P}$ with condition $P(0) = 10$

$f(P) = 0.10P$



* Slope Field of $\frac{dP}{dt} = 0.10P$ with condition $P(0) = 10$



Slopes: $\underline{t = 0}$

@ $P(0) = 10$, $\frac{dP}{dt} = 0.10(10) = 1$

@ $P(0) = 11$, $\frac{dP}{dt} = 0.10(11) = 1.1$

$\vdots$

@ $P(0) = P_0$, $\frac{dP}{dt} = 0.10(P_0)$

$\underline{t = 1}$

@ $P(1) = 10$, $\frac{dP}{dt} = 0.10(10) = 1$

@ $P(1) = 11$, $\frac{dP}{dt} = 0.10(11) = 1.1$

$\vdots$

@ $P(1) = P_1$, $\frac{dP}{dt} = 0.10(P_1)$