2nd-Order ODEs & Equilibriums

Objectives:

1. Introducing linear 2nd-order ODEs
2. Verifying solutions to linear 2nd-order ODEs
3. Equilibrium solutions to linear 2nd-order ODEs

Recall: Exponential Growth/Decay

$$\frac{dy}{dt} = ry \quad \leftarrow \text{1st-order, linear, homo, auto}$$

$\hookrightarrow$ r is a parameter

separation of variables

$$\int \frac{dy}{y} = \int r\,dt$$

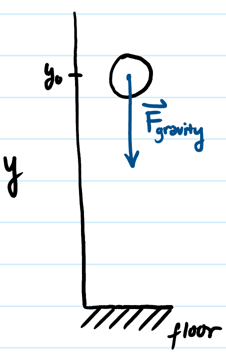
$$e^{\ln(y)} = e^{rt + C}$$

$$y(t) = e^{rt+C}$$
$$y(t) = Ae^{rt}$$

Verify: $\left.\begin{aligned} y &= Ae^{rt} \\ y' &= Are^{rt} \end{aligned}\right\}$ $\frac{dy}{dt} = ry$

$$A\cancel{r}\cancel{e^{rt}} = \cancel{rAe^{rt}}$$
$$1 = 1$$

Falling Object



Newton's 2nd Law of Motion: $\vec{F} = ma$

- $\vec{F}$ is the net force
- m is the mass of the object
- a is the acceleration

Assumptions:
- No air resistance
- gravity is the only force acting on the object (vertical fall)
- the weight is $\vec{F}$ because gravity is the only force and direction is downwards
- So, $\cancel{m}g = \cancel{m}a$

  $g = a \rightarrow$ gravity = acceleration regardless of mass
- Object is near earth surface

$\longrightarrow$ Let $y(t)$ be the height of the object,
$y'(t)$ be the velocity, and
$y''(t)$ be the acceleration.

Since y'' is the acceleration and $a = g$, then $\vec{F} = \vec{F}_{gravity}$ or $y'' = g$.

→ the 2nd-order ODE for modeling falling object is

$$y'' = g \leftarrow \text{2nd-order, linear, non-homo, auto}$$

→ Solution using direct integration: $\int y'' = \int g \, dt$

$$y' = gt + c_1 \leftarrow \text{velocity}$$

$$\int y' = \int (gt + c_1) dt$$

$$y(t) = \frac{gt^2}{2} + \underbrace{c_1 t + c_2}_{\text{two constants because we integrated twice.}} \leftarrow \text{height}$$

→ You need two initial conditions

$y(0) = y_0 \leftarrow$ initial height

$y'(0) = 0 \leftarrow$ initial velocity (held up at y_0 height)

→ Solution graph:

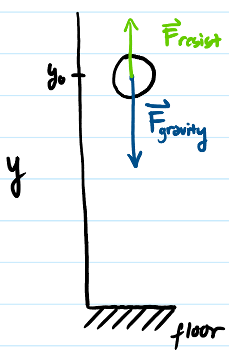
gravity is $g < 0$ (negative)

Earth $g = -9.8 \, m/s^2$

$$t_{floor} = \frac{c_1}{g} + \frac{\sqrt{2c_2 g + c_1^2}}{g} \quad \text{for } 0 = \frac{g}{2}t^2 + c_1 t + c_2$$

↑ time it takes to reach floor

Falling Object with Air Resistance



With the same assumptions at the Falling object example.

Additional assumptions:

- With air resistance (drag)
- Air resistance $\vec{F}_{resist}$ is the force acting against gravity which is proportional to the velocity.
- $\vec{F}_{resist} = -ky' \leftarrow$ for low velocity

  or

  $\vec{F}_{resist} = -k(y')^2 \leftarrow$ for high velocities

  ↳ k is the resistant coefficient

the total force is still $\vec{F} = ma$ or $\vec{F} = my''$.

$\longrightarrow$ ODE: $\vec{F} = \vec{F}_{gravity} + \vec{F}_{resist}$

$$\downarrow$$

$$my'' = g - ky'$$

or

$$my'' + ky' = g \longleftarrow \text{2nd-order, linear, auto, non-homo}$$

$$\downarrow$$

We can't solve this using direct integration but we can using other methods

Verifying Solutions to 2nd-order Linear ODEs

* Example 1: $y'' + 2y = 0 \longrightarrow$ proposed solutions $y_1 = \sin(\sqrt{2}\,t)$
$y_2 = \cos(\sqrt{2}\,t)$

- Verify $y_1 = \sin(\sqrt{2}\,t)$
$y_1' = \sqrt{2}\cos(\sqrt{2}\,t)$
$y_1'' = -2\sin(\sqrt{2}\,t)$

$$y'' + 2y = 0 \longrightarrow -2\cancel{\sin(\sqrt{2}\,t)} + 2\cancel{\sin(\sqrt{2}\,t)} = 0$$

$0 = 0$ ✓ So, y_1 is a solution.

- Verify $y_2 = \cos(\sqrt{2}\,t)$
$y_2' = -\sqrt{2}\sin(\sqrt{2}\,t)$
$y_2'' = -4\cos(\sqrt{2}\,t)$

$$y'' + 2y = 0 \longrightarrow -2\cancel{\cos(\sqrt{2}\,t)} + 2\cancel{\cos(\sqrt{2}\,t)} = 0$$

$0 = 0$ So, y_2 is a solution

* Example 2: $y'' - y = 0 \longrightarrow$ proposed solutions $y_1 = e^{t}$
$y_2 = e^{-t}$

- Verify $y_1 = e^{t}$
$y_1' = e^{t}$
$y_1'' = e^{t}$

$$y'' - y = 0 \longrightarrow \cancel{e^{t}} - \cancel{e^{t}} = 0$$

$0 = 0$ So, y_1 is a solution.

- Verify $y_2 = e^{-t}$
$y_2' = -e^{-t}$
$y_2'' = e^{-t}$

$$y'' - y = 0 \longrightarrow \cancel{e^{-t}} - \cancel{e^{-t}} = 0$$

$0 = 0$ So, y_2 is a solution

$$y''-y=0 \longrightarrow \cancel{e^{-t}}-\cancel{e^{-t}}=0$$

$0=0$ So, y_2 is a solution.

* Example 3: $y''+3y'=0 \longrightarrow$ proposed solution $y=e^{-3t}$

- verify $y=e^{-3t}$
  $y'=-3e^{-3t}$
  $y''=9e^{-3t}$

$$y''+3y'=0 \longrightarrow 9e^{-3t}+3(-3e^{-3t})=0$$

$$\cancel{9e^{-3t}}-\cancel{9e^{-3t}}=0$$

$$0=0$$

<u>Equilibrium Solutions of 2nd-Order Linear ODEs</u>

An <u>equilibrium solution</u> is a constant solution y where $y'=0$ and $y''=0$.

* Example 4: $y''-y=0 \longrightarrow$ set $y''=0$.

$0-y=0 \longrightarrow y=0$ a constant
So, an equilibrium solution.

* Example 5: $y''+3y'=0 \longrightarrow$ set $y'=0$ & $y''=0$.

$0+3(0)=0 \longrightarrow 0=0$, meaning an equilibrium exists but we can't know it... yet.