Analyzing Equilibriums of a System of 1st-Order ODEs

Objectives:

1. Introduce the "straight-line" solutions for linear systems of 1st-order ODEs.
2. Viewing the "straight-line" solutions on a phase plane.
3. Analyzing the behavior of solutions around an equilibrium (stability).

Recall: Horizontal Spring-Mass System

* Assuming a mass of 1:

$x'' + bx' + kx = 0$ ← no external forces

↑ friction coeff b ; ↑ spring constant k

* Converted into a system of 1st-order ODEs

$x' = v$ ← velocity

$v' = -kx - bv$ ← acceleration

Recall: Isoclines

The slope of direction vectors is $m = \frac{v'}{x'}$, where m is a constant.

Then, on the line $m = \frac{-kx - bv}{v} \longrightarrow v = -\frac{k}{m+b}x$ on the phase plane,

the direction vectors have the same slope of m.

"Straight-line" Solutions

* A "straight-line" solution is a special case of isocline where the direction vectors on the phase plane is parallel to the line.

* The ratio $\frac{v}{x}$ is a constant. That is, $\frac{v}{x} = m$, where $m = \frac{v'}{x'}$.

This means that the ratio of x and v is changing proportionally by a constant $m = \frac{v'}{x'}$ (isocline).

Determining the "straight-line" Solutions

* Example 1: $b = 3$; $k = 2$;
$$x' = v$$
$$v' = -2x - 3v$$

Let $\frac{v'}{x'} = \frac{v}{x}$.

$$\frac{-2x - 3v}{v} = \frac{v}{x}$$
$$(-2x - 3v)x = v^2$$
$$-2x^2 - 3vx = v^2$$
$$0 = v^2 + 3vx + 2x^2$$

↓ solve for v

$$0 = (v + x)(v + 2x) \longrightarrow v_1 = -2x, \quad v_2 = -x$$

or

$$v = \frac{-3x \pm \sqrt{(3x)^2 - 4(1)(2x^2)}}{2(1)}$$

$$= -\frac{3}{2}x \pm \frac{1}{2}x$$

$$v = \left(-\frac{3}{2} \pm \frac{1}{2}\right)x \longrightarrow \begin{array}{l} v_1 = -2x \\ v_2 = -x \end{array}$$

So, the two "straight-lines" on the phase plane is

$$v_1 = -2x_1 \text{ and } v_2 = -x_2.$$

Now, for the solutions:

- $v_1 = -2x_1$:

$$x_1' = v_1$$
$$v_1' = -2x_1 - 3v_1$$

plug-in $v_1 = -2x_1$

$$x_1' = -2x_1 \xrightarrow{\text{SOV}} \int \frac{dx_1}{x_1} = \int -2\,dt$$

$$e^{\ln(x_1)} = e^{-2t + c}$$

$$x_1 = c_1 e^{-2t}$$

$$v_1' = -2\left(-\frac{1}{2}v_1\right) - 3v_1$$
$$= v_1 - 3v_1$$
$$v' = -2v_1 \xrightarrow{\text{SOV}} \int \frac{dv_1}{v_1} = \int -2\,dt$$

$$e^{\ln(v_1)} = e^{-2t + C}$$

$$v_1 = c_1 e^{-2t}$$

let $c_1 \longrightarrow -2c_1$.

$$v_1 = -2\boxed{c_1 e^{-2t}} \longleftarrow x_1$$

So, the "straight-line" solutions on the line $v_1 = -2x_1$ are

$$\left.\begin{array}{l} x_1 = c_1 e^{\boxed{-2}t} \\ v_1 = -2c_1 e^{\boxed{-2}t} \end{array}\right\} \text{the exponent } -2 \text{ is called an eigenvalue.}$$

- $v_2 = -x_2$:

$$x_2' = v_2$$
$$v_2' = -2x_2 - 3v_2$$

plug in $v_2 = -x_2$

$$x_2' = -x_2 \xrightarrow{\text{SOV}} \int \frac{dx_2}{x_2} = \int -1\,dt$$

$$e^{\ln(x_2)} = e^{-t + C}$$

$$x_2 = c_2 e^{-t}$$

$$v_2' = -2(-v_2) - 3v_2$$
$$= 2v_2 - 3v_2$$
$$v_2' = -v_2 \xrightarrow{\text{SOV}} \int \frac{dv_2}{v_2} = \int -1\,dt$$

$$e^{\ln(v_2)} = e^{-t + C}$$

$$v_2 = c_2 e^{-t}$$

let $c_2 \longrightarrow -c_2$

$$v_2 = -\boxed{c_2 e^{-t}} \longleftarrow x_2$$

So, the "straight-line" solutions on the line $v_1 = -x_1$ are

$$x_2 = c_2 e^{\boxed{-1}t}$$
$$v_2 = -c_2 e^{\boxed{-1}t}$$

} the exponent -1 is called an eigenvalue.

In summary, there are two "straight-line" solutions corresponding to two eigenvalues (exponents of the solutions).

By the superposition principle, the general solution is

$$x = c_1 x_1 + c_2 x_2$$
$$v = c_1 v_1 + c_2 v_2$$

"absorb" constants

$$x = c_1 e^{-2t} + c_2 e^{-t}$$
$$v = -2c_1 e^{-2t} - c_2 e^{-t}$$

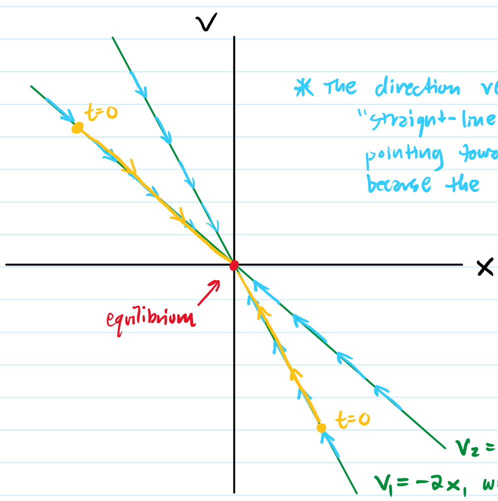
1st "straight-line" solution — 2nd "straight-line" solution

check → Since $x' = v$, then

$$x = c_1 e^{-2t} + c_2 e^{-t}$$
$$x' = \underbrace{-2c_1 e^{-2t} - c_2 e^{-t}}_{v} \quad \checkmark$$

- The "straight-line" solutions on the phase plane.

* Specific solution curves appear to be a straight line on the phase plane.



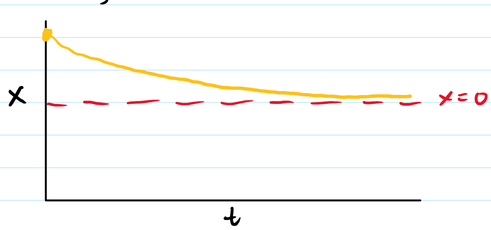
* The direction vectors on these "straight-line" solutions are pointing towards the equilibrium because the eigenvalues are negative.

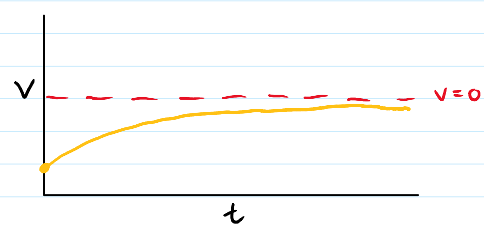
$v_2 = -x_1$, with -1 eigenvalue

$v_1 = -2x_1$, with -2 eigenvalue

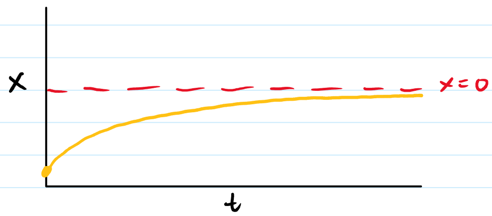
* When viewed on the t-x and t-v plane, they are exponential curves.

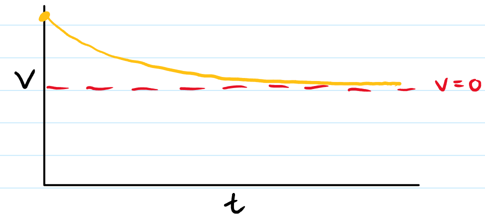
- $v_1 = -2x_1$; e^{-2t} solution





- $v_2 = -x_2$; e^{-t} solution





Eigenvalues

* Eigenvalues are the value of the exponents of the "straight-line" solutions. It also tells how fast the solution decays/grows.

Example: From the previous example the two "straight-line" solutions are

$$e^{\boxed{-2}t} \quad \text{and} \quad e^{\boxed{-}t}.$$

the eigenvalues are $\boxed{-2}$ and $\boxed{-1}$ respectively.
Since both eigenvalue are negative real numbers,
the equilibrium stability is a stable node.

both negative — real numbers

Stability of Equilibriums

* the sign of the eigenvalues (exponents of the "straight-line" solutions) can determine the behavior of solutions around an equilibrium (stability).

* Let λ_1 and λ_2 be the eigenvalues.

Types of equilibrium stability for a linear system of 1st-order ODEs.

- node
  - stable if λ_1 & λ_2 are negative real eigenvalues
  - unstable if λ_1 & λ_2 are positive real eigenvalues
  - If $\lambda_1 = \lambda_2$, then node is improper or star
- saddle if λ_1 & λ_2 are opposite signs real eigenvalues
- spiral
  - stable if λ_1 & λ_2 are complex numbered eigenvalues with negative real part
  - unstable if λ_1 & λ_2 are complex numbered eigenvalues with positive real part
- center if λ_1 & λ_2 are purely imaginary numbers
- Fixed points
  - stable if One of the eigenvalues are 0 and the other is negative real number
  - unstable if One of the eigenvalues are 0 and the other is positive real number