Applications of 1st-Order ODEs

Objectives:
1. Summarize ODE models in application problems

Recall: Classification of Equations
- Order
- Autonomy
- Homogeneity
- Linearity

Exponential Growth/Decay Model

$\frac{dy}{dt} = ky$, $y(0) = y_0$ ⟶ 1st-order, auto, homo, linear

k is the parameter: If $k < 0$, decay
If $k > 0$, growth

General Solution: $y(t) = C_1 e^{kt}$ ⟵ Separation of Variables

* Modifications

⟶ Radioactive decay: $\frac{dy}{dt} = -ry$, $y(0) = y_0$, $r > 0$

r is the decay constant

Terms:
- $y(t)$ is the amount of molecules at time t
- $-ry$ is the rate of decay.

General Solution: $y(t) = C_1 e^{-rt}$ ⟵ Separation of Variables

⟶ Newton's Law of Cooling: $\frac{dT}{dt} = k(T - T_A)$, $T(0) = T_0$, $k < 0$

k is the constant of proportionality
T_A is the ambient temperature

Terms:
- $T(t)$ is the temperature at time t.
- $(T - T_A)$ is the difference in the current temperature to the ambient temperature.

General Solution: $T(t) = T_A + C_1 e^{-kt}$ ⟵ Separation of Variables

⟶ Population growth: $\frac{dy}{dt} = ky$, $y(0) = y_0$, $k > 0$

k is the intrinsic growth rate

Terms:
- $y(t)$ is the population at time t.
- ky is the growth rate

General Solution: $y(t) = C_1 e^{kt}$ ⟵ Separation of Variables

⟶ Logistic Growth/Decay: $\frac{dy}{dt} = ky - \frac{ky^2}{N}$, $y(0) = y_0$ ⟶ 1st-order, auto, homo, non-linear

k is the intrinsic growth rate
N is the carrying capacity

or

$\frac{dy}{dt} = ky(N - y)$

Terms:
- $y(t)$ is the population at time t.
- ky is the natural growth rate.
- $-\frac{ky^2}{N}$ is the rate of decrease due to environmental pressure.

General Solution: $y(t) = \dfrac{C_1 N e^{kt}}{1 + C_1 e^{kt}}$ ⟵ Separation of Variables

Mixed Growth and Decay

rate of change = rate of increase − rate of decrease ⟵ General idea

* **Modifications**

⟶ Radioactive production and decay : $\dfrac{dy}{dt} = a - ky$, $y(0) = y_0$, $k > 0$, $a > 0$ ⟵ 1st-order, auto, non-homo, linear

- a is the rate of production
- k is the decay constant

Terms:
- $y(t)$ is the amount of molecules at time t
- a is the rate of production
- $-ky$ is the decay rate

General Solution: $y(t) = \dfrac{a}{k} + C_1 e^{-kt}$ ⟵ Separation of Variables

⟶ Mixing Problem: rate of change = flow-in − flow-out ⟵ general idea

$$\frac{dy}{dt} = a - \frac{\beta y}{V(t)}, \quad y(0) = y_0, \quad a \geq 0, \quad 0 < \beta < 1$$

⟵ 1st-order, linear, homo (if $a = 0$) non-homo (if $a > 0$), auto (if $V(t)$ is constant) non-auto (if $V(t)$ is non-constant)

- a is the flow-in rate
- β is the intrinsic flow-out rate (like a filter)
- $V(t)$ is the volume at time t.

Terms:
- $y(t)$ is the amount of solution concentration at time t.
- a is the constant flow-in rate
- $-\dfrac{\beta y}{V(t)}$ is the flow-out rate depending on out the volume changes in t.

General solution: See integrating factors method