Bifurcations in One Dimension

Objectives:

1. Introduce bifurcations
2. Bifurcation analysis

Example 1: Exponential growth/decay

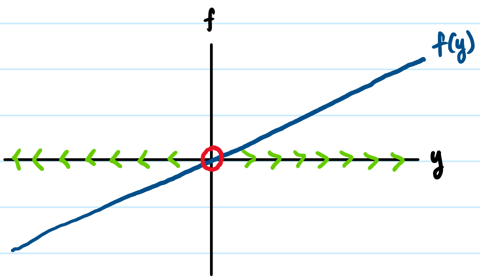
$$\frac{dy}{dt} = \underbrace{\boxed{ky}}_{f(y)}, \text{ where } k \in (-\infty, \infty) \text{ is a parameter}$$

- We know that $f(y) < 0$ if $k < 0$,
  $f(y) > 0$ if $k > 0$, and
  $f(y) = 0$ if $k = 0$.

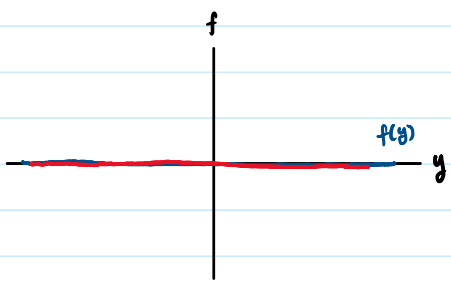
- Equilibrium:
  - critical points; $0 = ky \longrightarrow y^* = 0$
  - Equilibrium solution $\longrightarrow y(t) = y^* = 0$

- Stability:
  - $\frac{\partial f}{\partial y} = k \longrightarrow y(t) = 0$ is stable if $k < 0$,
    unstable if $k > 0$, and
    unknown if $k = 0$.
  - If $\frac{\partial f}{\partial y} = 0$, then $\frac{\partial^2 f}{\partial y^2} = 0 \longrightarrow$ not a semi-stable equilibrium

- Phase diagram for $k > 0$:



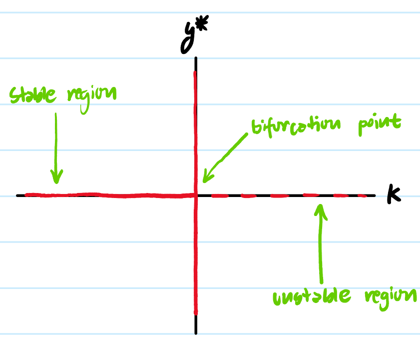
- Phase diagram for $k = 0$:



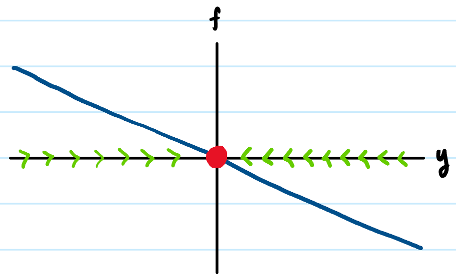
- Phase diagram for $k < 0$:

- In this example, the bifurcation is the change in equilibrium stability.

- Bifurcation diagram



- The bifurcation point is at $k = 0$, where the stability changes.



Example 2:

$$\frac{dy}{dt} = \underbrace{\boxed{y^2 + b}}_{f(y)}, \text{ where } b \in (-\infty, \infty) \text{ is a parameter}$$

- Equilibriums:
  - critical points; $0 = y^2 + b \longrightarrow y = \pm\sqrt{-b}$

    $y_1^* = -\sqrt{-b}$, $y_2^* = +\sqrt{-b}$
  - Equilibrium solutions $\longrightarrow$ $y(t) = -\sqrt{-b}$

    $y(t) = +\sqrt{-b}$
    - $\longrightarrow$ If $b > 0$, then there are no equilibriums.
    - $\longrightarrow$ If $b = 0$, then there is only one equilibrium.
    - $\longrightarrow$ If $b < 0$, then there are two equilibriums

- Stability:
  - $\frac{\partial f}{\partial y} = 2y$
  - If $b = 0$, then the equilibrium is $y(t) = 0$.

    $\left.\frac{\partial f}{\partial y}\right|_{y=0} = 0 \longrightarrow$ semi-stable (maybe).

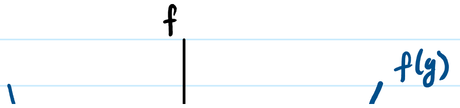
    $\frac{\partial^2 f}{\partial y^2} = 2 \longrightarrow \left.\frac{\partial^2 f}{\partial y^2}\right|_{y=0} > 0 \longrightarrow$ increasing, $\frac{dy}{dt} > 0$

    So, solutions around $y(t) = 0$ is increasing.
  - If $b < 0$, then the equilibriums are $y(t) = -\sqrt{-b}$ and $y(t) = +\sqrt{-b}$.

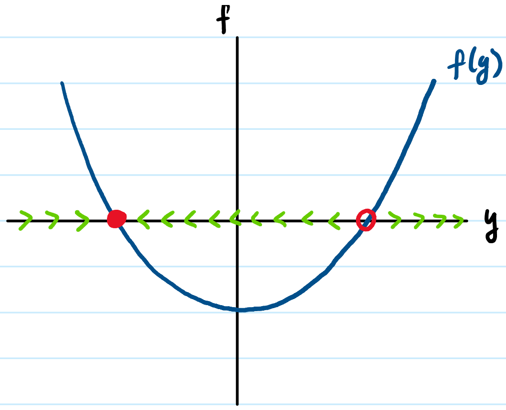
    $\left.\frac{\partial f}{\partial y}\right|_{y=-\sqrt{-b}} = -2\sqrt{-b} < 0 \longrightarrow y(t) = -\sqrt{-b}$ is stable.

    $\left.\frac{\partial f}{\partial y}\right|_{y=+\sqrt{-b}} = 2\sqrt{-b} > 0 \longrightarrow y(t) = +\sqrt{-b}$ is unstable.

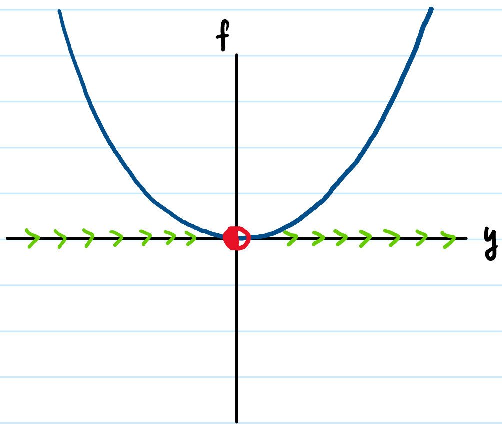
- Phase diagram for $b < 0$:



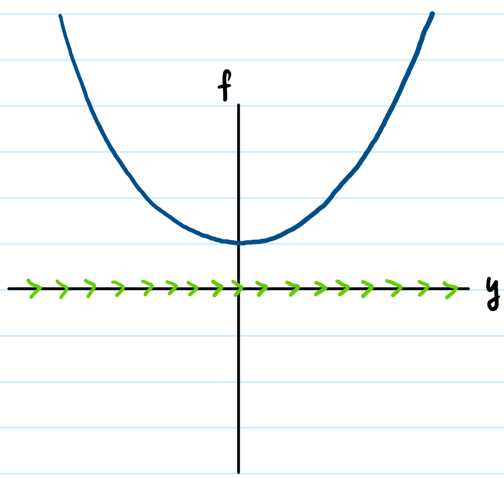
- In this example, the bifurcation is the change in the number of equilibriums.



- Phase diagram for $b = 0$:

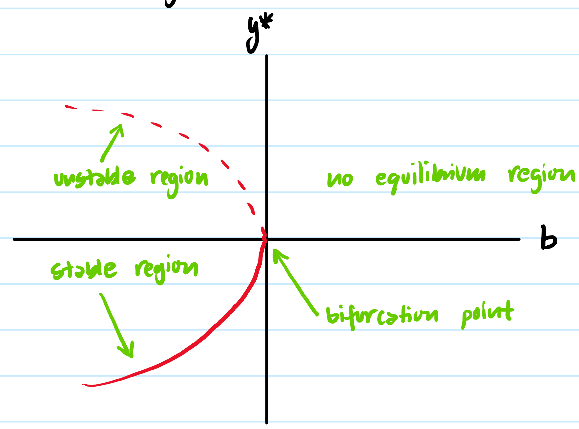


- Phase diagram for $b > 0$:



is the change in the number of equilibriums.

- Bifurcation diagram



- The <u>bifurcation point</u> is at $b = 0$, where the number of equilibrium points changes.

<u>Definitions</u>:
- A <u>bifurcation</u> is the change in the number of equilibriums or the stability of equilibriums as a parameter changes.
- A <u>bifurcation point</u> is the parameter value on which the change has occured.