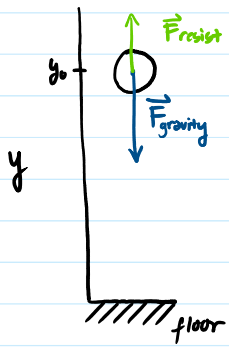
Characteristic Polynomials for Homogeneous Linear ODEs

Objectives:

- Introduce on how to determine the homogeneous solution.
- Creating characteristic equations.
- Solving linear homogeneous 2nd-order ODEs using the roots of the characteristic equation.

Recall: Falling Object with Air resistance



- Total force: $\vec{F} = ma$ or $\vec{F} = my''$
- Force due to gravity: $\vec{F}_{grav} = mg$, $g < 0$
- Force due to air resistance: $\vec{F}_{resist} = -ky'$, $k > 0$

$$\vec{F} = \vec{F}_{grav} + \vec{F}_{resist}$$

$$my'' = mg - ky'$$

$$y'' = g - \frac{k}{m}y' \quad \text{or} \quad y'' + \frac{k}{m}y' - g = 0 \longleftarrow \text{2nd-order, linear, auto, homo}$$

Recall: Solutions to 2nd-Order linear ODEs

* $y'' + py' + qy = f(t)$; $p, q, f(t)$ are functions of t.

- If $f(t) = 0$, then the ODE is homo.
- If $f(t) \neq 0$, then the ODE is non-homo.

* Solutions: $y = y_h + y_p$ ← particular solution

↑ homogeneous solution

- If $f(t) = 0$, then $y_p = 0$.
- If $f(t) \neq 0$, then y_p is a function that satisfies the ODE.

Recall: Exponential Growth/Decay

* $y' = ky$, for some constant k

↓ separation of variables

$y(t) = Ae^{kt}$ ← exponential

* Verification Process

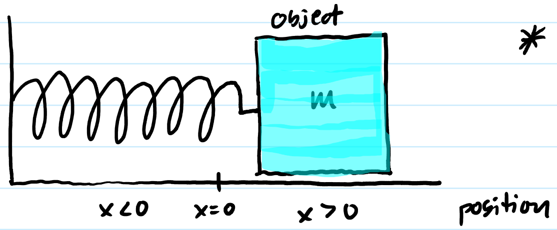
$y = Ae^{kt}$

$y' = Ake^{kt}$

$y' = ky \longrightarrow Ake^{kt} = kAe^{kt} \longrightarrow 1 = 1$ ✓

Spring Mass System

* Explore Newton's Law of Motion $F = ma$.



* forces acting on the system
  - spring
  - mass
  - friction
  - normal force is "cancelled" out due to gravity because object is horizontal.

→ Newton's 2nd Law of Motion

force = mass · acceleration

$$\vec{F} = ma$$

$$\downarrow$$

$$\vec{F} = m x''$$ ← acceleration (2nd derivative of a position function $x(t)$)

→ Hooke's Law for extending or compressing a spring

$F_{spring} = -kx$ → where k is the spring constant

→ Friction equation

$F_{friction} = -bx'$ → where b is the proportional constant

velocity (1st derivative of $x(t)$)

* Governing ODE with constant coefficients

→ Let $x(t)$ be the position the object at time t.

$$\vec{F} = \vec{F}_{friction} + \vec{F}_{spring} + \vec{F}_{external}$$

$$\vec{F} - \vec{F}_{friction} - \vec{F}_{spring} = \vec{F}_{external}$$

$$\downarrow$$

$$mx'' - (-bx') - (-kx) = 0$$ ← assuming no external forces

$$\downarrow$$

$$mx'' + bx' + kx = 0$$

or

$$x'' + \frac{b}{m}x' + \frac{k}{m}x = 0$$

* To simplify the ODE, we let $m = 1$.

$$x'' + bx' + kx = 0$$ ← 2nd-order homogeneous linear ODE

$$x'' + bx' + kx = 0$$ ← 2nd-order homogeneous linear ODE with constant coefficients

We want to find $x(t)$ that satisfies the ODE. $x(t)$ is the homogeneous solution.

Characteristic Equation

$$x'' + bx' + kx = 0$$ ← 2nd-order, homo, linear, auto with constant coefficients

Let $x = e^{rt}$ for some r constant.

trial solution (Ansatz)

$$x' = re^{rt}$$
$$x'' = r^2 e^{rt}$$

$$x'' + bx' + kx = 0$$
$$r^2 e^{rt} + bre^{rt} + ke^{rt} = 0$$
$$e^{rt}\underbrace{(r^2 + br + k)}_{0} = 0, \quad e^{rt} \neq 0$$

$$r^2 + br + k = 0$$ ← characteristic equation

the roots r determines the homogeneous solution

Roots of the Characteristic Equation

- If r is distinct real roots r_1 & r_2, then

$$x = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

- If r is a repeated real root $r = r_1 = r_2$, then

$$x = C_1 e^{rt} + C_2 t e^{rt}$$

- If r is a complex conjugate root $r = \alpha + \beta i$ ← imaginary part

real part

$$x = e^{\alpha t}\left(C_1 \cos(\beta t) + C_2 \sin(\beta t)\right)$$

Horizontal Spring-Mass Model

$$x'' + bx' + kx = 0$$

↳ spring constant

friction coeff

spring constant

friction coeff

- Example: $x'' + 6x' + 4x = 0$

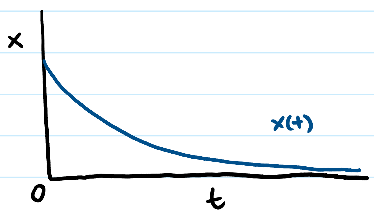
$$\downarrow$$

$$r^2 + 6r + 4 = 0$$

$$\downarrow$$

$$r = -3 \pm \sqrt{5} \quad \begin{cases} r_1 = -3 - \sqrt{5} \\ r_2 = -3 + \sqrt{5} \end{cases}$$ distinct real roots

$$x(t) = C_1 e^{(-3-\sqrt{5})t} + C_2 e^{(-3+\sqrt{5})t} \longleftarrow$$ overdamped



- Example: $x'' + 4x' + 4x = 0$

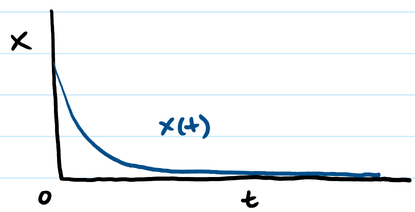
$$\downarrow$$

$$r^2 + 4r + 4 = 0$$

$$\downarrow$$

$$r = -2 \text{ (mult. of 2)} \longleftarrow$$ repeated real root

$$x(t) = C_1 e^{-2t} + C_2 t e^{-2t} \longleftarrow$$ critically damped



- Example: $x'' + x' + 4x = 0$

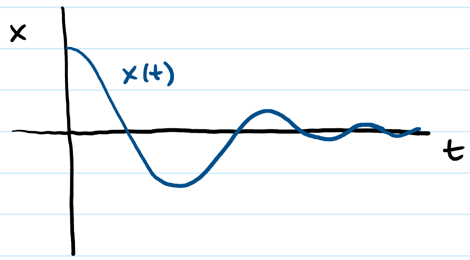
$$\downarrow$$

$$r^2 + r + 4 = 0$$

$$\downarrow$$

$$r = -\underbrace{\frac{1}{2}}_{\alpha = -1/2} \pm i \underbrace{\frac{\sqrt{15}}{2}}_{\beta = \sqrt{15}/2} \longleftarrow$$ complex conjugate roots

$$x(t) = e^{-1/2 t}\left(C_1 \cos\left(\sqrt{15}/2\, t\right) + C_2 \sin\left(\sqrt{15}/2\, t\right)\right) \longleftarrow$$ underdamped



- Example: $x'' + 4x = 0$

$$\downarrow$$

$$r^2 + 4 = 0$$

$$\downarrow$$

$$r = \pm 2i$$ ← complex conjugate root (purely imaginary)

$\alpha = 0$ $\beta = 2$

$$x(t) = C_1 \cos(2t) + C_2 \sin(2t)$$ ← undamped