# Classification of Equations & Principles of Solutions

Objectives:

1. Layout the different classes of ODEs.
2. Explain the general and specific solution.
3. Show how to verify a solution.

## Classification of Equations

0. Ordinary vs. Partial: ODEs have one independent variables.
   PDEs have two or more independent variables.

Examples:

| Ordinary | Partial |
|---|---|
| $\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0$ | $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$ |

1. Order: the highest derivative present - order, Let $y' = \frac{dy}{dt}$, $y'' = \frac{d^2y}{dt^2}$, and $y^{(m)} = \frac{d^my}{dt^m}$

Examples:

1st order $\rightarrow P_1(t)y' + P_0(t)y = G(t)$

2nd order $\rightarrow P_2(t)y'' + P_1(t)y' + P_0(t)y = G(t)$

$\vdots$

nth order $\rightarrow P_n(t)y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = G(t)$

2. Autonomy: whether or not there is a function of $t$ in the rules of ODE.
   - Autonomous: An ODE which is not an explicit function of $t$.
   - Non-Autonomous: An ODE which is an explicit function of $t$.

Examples:

| Auto | non-Auto |
|---|---|
| $y' = ky$ | $y' = kyt$ |
| $y' = ky(1-\frac{y}{N})$ | $y' = t$ |

3. Homogeneity: the RHS of the ODE may be zero or not.
   - Homogeneous: The RHS is zero.
   - Non-Homogeneous: the RHS is non-zero.

Examples:

| Homo | Non-Homo |
|---|---|
| $y'' + y = 0$ | $y'' + y = t$ |
| $y' + t^2y = 0$ | $y' + t^2y = t + 2t$ |
| $y''' + ty'' + y = 0$ | $y'' + ty'' + y = 6t + 3$ |

4. Linearity: When the derivatives and variables in the ODE are multiplied by constants or functions of $t \rightarrow$ linear in the dependent variable.
   In other words, the ODE can be written in the form

$$P_n(t)\frac{d^ny}{dt^n} + P_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_2(t)\frac{d^2y}{dt^2} + P_1(t)\frac{dy}{dt} + P_0(t)y = G(t).$$

Examples: linear | Non-linear

| linear | Non-linear |
| --- | --- |
| $y'' + y = 0$ | $y' + 1/y = 0$ |
| $y'' + 2y' + y = 1$ | $y' + y^2 = 0$ |
| $\sin(t)y' + \cos(t)y = e^t$ | $y'' + \sin(y) = 0$ |

## Types of Exact Solution

An exact solution is a mathematical expression solved analytically (e.g. using calculus) that satisfies the ODE.
⟶ there are two types: (1) general and (2) specific.

* Example: Exponential growth/decay

$\frac{dy}{dt} = ky$ where $k$ is some constant. ⟶ 1st-order, linear, auto, Homo

↓ general solution

$y(t) = Ae^{kt}$ where $A$ is an arbitray constant.

↓ Specific solution if $\underbrace{y(0) = y_0}_{\text{initial condition}}$ is given.

Given $y(0) = y_0$,

$$y(0) = Ae^{k(0)}$$
$$y_0 = A.$$

So, $y(t) = y_0 e^{kt}$ is the specific solution.

## What makes a solution valid? (Verifying Solutions)

* Example: solution ⟶ $y(t) = Ae^{kt}$, ODE: $\frac{dy}{dt} = ky$

↓ compute $dy/dt$

$$\frac{dy}{dt} = kAe^{kt}$$

↓ substitute $y$ and $\frac{dy}{dt}$ into ODE

$$\frac{dy}{dt} = ky$$
$$\cancel{kAe^{kt}} = \cancel{k}\cancel{A}\cancel{e^{kt}}$$
$$1 = 1$$ ⟶ identity

Thus, $y(t) = Ae^{kt}$ is the solution of $\frac{dy}{dt} = ky$.

## More Verifying Solutions

* Example: $\frac{dy}{dt} = 3t^2(1+y)$ ⟶ 1st-order, linear, non-auto, non-homo

* Example: $\frac{dy}{dt} = 3t^2(1+y)$ $\longrightarrow$ 1st order, linear, non-zero, non-homo

$$\Big\downarrow \text{solution}$$

$y(t) = -1 + ke^{t^3}, \; k \neq 0$ $\longrightarrow$ general solution

Verification: $\frac{dy}{dt} = 0 + ke^{t^3} 3t^2 = 3kt^2e^{t^3}$

$$\frac{dy}{dt} = 3t^2(1+y)$$

$$3kt^2e^{t^3} = 3t^2(\cancel{1} + \cancel{(-1} + ke^{t^3}))$$

$$3kt^2e^{t^3} = 3kt^2e^{t^3} \quad \checkmark$$