Euler's Method

Objectives:

1. Introduce Euler's method

Recall: <u>Solving 1st-order ODEs.</u>

- Graphical: → Phase diagrams
  → Slope fields
- Analytical: → Separation of variables
  → Integrating factors

Another way of solving an ODE: Numerically using Euler's method.

Recall: <u>Limit Definition of the derivative</u>.

Let $y(t)$ be a continuous function of t.

$$\frac{dy}{dt} = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}$$

or

$$\frac{dy}{dt} = \lim_{h \to 0} \frac{y_{t+h} - y_t}{h}$$

<u>Euler's Method Set-up</u>

Let $\frac{dy}{dt} = f(t,y)$

$$\downarrow$$

$$\lim_{h \to 0} \frac{y_{t+h} - y_t}{h} = f(t,y)$$

$$\underbrace{y_{t+h}}_{\text{next value}} = \underbrace{y_t}_{\text{current value}} + \underbrace{h}_{\text{step size}}\underbrace{f(t,y)}_{\text{slope}}$$

<u>Approximate a specific solution using Euler's Method</u>

Suppose we have an IVP.

$$\frac{dy}{dt} = f(t,y), \quad y(t_0) = y_0$$

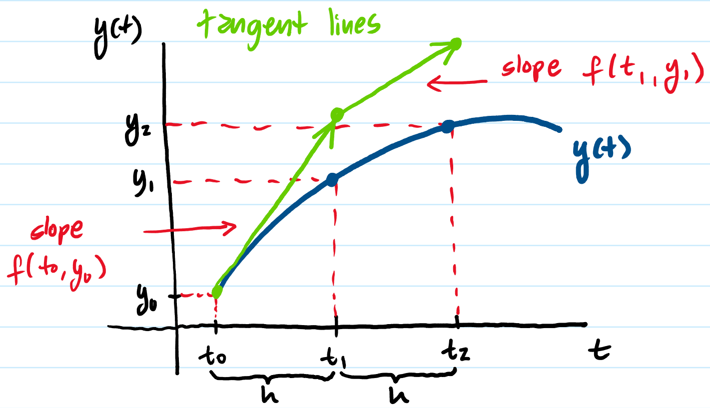
@ the initial condition: $\left.\frac{dy}{dt}\right|_{t=t_0} = f(t_0, y_0)$

Equation of the tangent line at $t=t_0$: $y = y_0 + f(t_0,y_0)(t-t_0)$

@ $t=t_1$: $y_1 = y_0 + f(t_0,y_0)(t_1-t_0)$
@ $t=t_2$: $y_2 = y_1 + f(t_1,y_1)(t_2-t_1)$
@ $t=t_3$: $y_3 = y_2 + f(t_2,y_2)(t_3-t_2)$
$\vdots$
@ $t=t_{n+1}$: $y_{n+1} = y_n + f(t_n,y_n)(t_{n+1}-t_n)$

Let $f_n = f(t_n,y_n)$ and $h = t_{n+1}-t_n$, → step size
then

$$y_{n+1} = y_n + f_n h, \quad f_0 = f(t_0,y_0).$$



This is called the Euler's method or the "tip-to-tail" method.

- Example OD: $\dfrac{dy}{dt} = t-y \longrightarrow f(t,y) = t-y$

  $y(0) = 0 \longrightarrow t_0 = 0$ & $y_0 = 0$

Suppose step size $h = 1/2$.

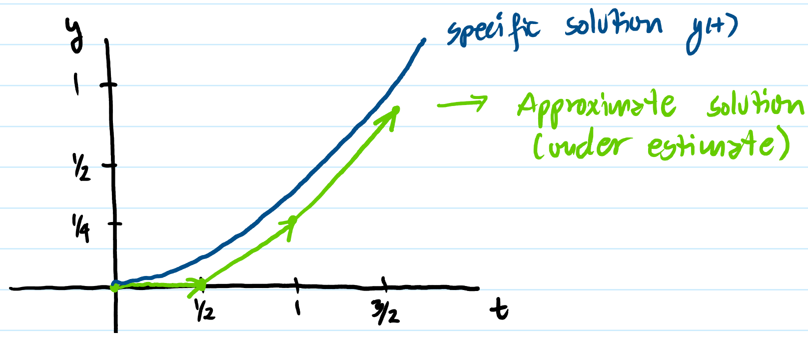
Euler's Formula $y_{n+1} = y_n + f(t_n,y_n)h$, $t_{n+1} = t_n + h$

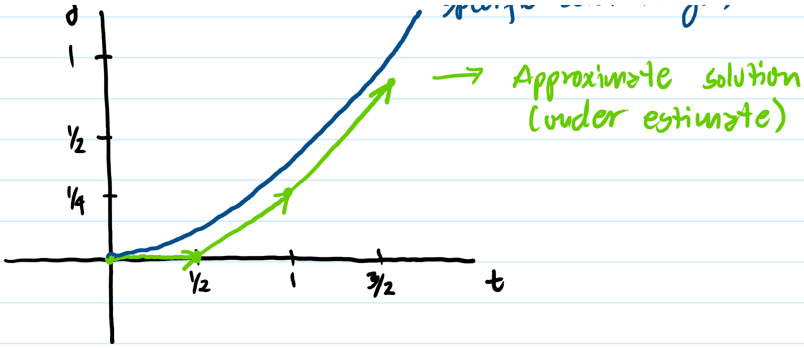
$n=0$: $t=t_1$, $y_1 = y_0 + (t_n - y_n)(1/2) = 0 + (0-0)(1/2) = 0$, $t_1 = 0 + 1/2 = 1/2$

$n=1$: $t=t_2$, $y_2 = y_1 + (t_1 - y_1)(1/2) = 0 + (1/2 - 0)(1/2) = 1/4$, $t_2 = t_1 + 1/2 = 1/2 + 1/2 = 1$

$n=2$: $t=t_3$, $y_3 = y_2 + (t_2 - y_2)(1/2) = 1/4 + (1 - 1/4)(1/2) = 5/8$, $t_3 = t_2 + 1/2 = 1 + 1/2 = 3/2$

$\vdots$





As a table:

n	t_n	y_n	$y_{n+1} = y_n + hf$
0	0	0	$\boxed{0 + (1/2)(0-0)}$
1	1/2	0	$\boxed{0 + (1/2)(1/2 - 0)}$
2	1	1/4	$\boxed{1/4 + (1/2)(1 - 1/4)}$
3	1.5	5/8	$\boxed{5/8 + (1/2)(1.5 - 5/8)}$
4	2	*	⋮