Existence and Uniqueness & Direction Fields and Phase Portraits

Objectives

1. Introduce direction fields
2. Introduce phase portraits
3. Existence and uniqueness of systems of 1st-order ODEs.

Recall: Horizontal Spring-Mass System

* Assuming a mass of 1:

$$x'' + bx' + kx = 0$$ ⟵ no external forces

↑ friction coeff b ; ↑ spring constant k

* Converted into a system of 1st-order ODEs

$x' = v$ ⟵ velocity
$v' = -kx - bv$ ⟵ acceleration

* Nullclines

- x-nullcline: $x' = 0 \rightarrow v = 0 \rightarrow$ no change in x
  $v' = -kx$
  - If $x < 0$, then $v' > 0$. ↑
  - If $x > 0$, then $v' < 0$. ↓
- v-nullcline: $0 = -kx - bv$
  $bv = -kx$
  $v = -\frac{k}{b}x \rightarrow$ no change in v
  $x' = v$
  - If $v < 0$, then $x' < 0$. ⟵
  - If $v > 0$, then $x' > 0$. ⟶

Isoclines

* Isoclines are lines on which direction vectors have the same slope.
  Nullclines are special case of isoclines with slope zero.

Example: $x' = v$
$v' = -kx - bv$

When viewed on a x-v axis, let $\frac{v'}{x'} = m$, for some m constant.
↑ vector slope

So,
$$\frac{v'}{x'} = \frac{-kx - bv}{v}$$
$$m = \frac{-kx - bv}{v}$$
$$mv = -kx - bv$$
$$mv + bv = -kx$$
$$v(m+b) = -kx$$
$$v = \frac{-k}{m+b}x$$ ⟵ isocline

→ Nullclines:

$$\lim_{m \to 0} \frac{-k}{m+b}x = \frac{-k}{b}x$$ directions ⟵ ⟶ ⟵ v-nullcline $v' = 0$

$$\lim_{m \to \infty} \frac{-k}{m+b}x = 0$$ directions ↑ ↓ ⟵ x-nullcline $x' = 0$

$$\lim_{m \to -b} \frac{-k}{m+b}x = \text{DNE}$$ ⟵ the x-nullcline is vertical

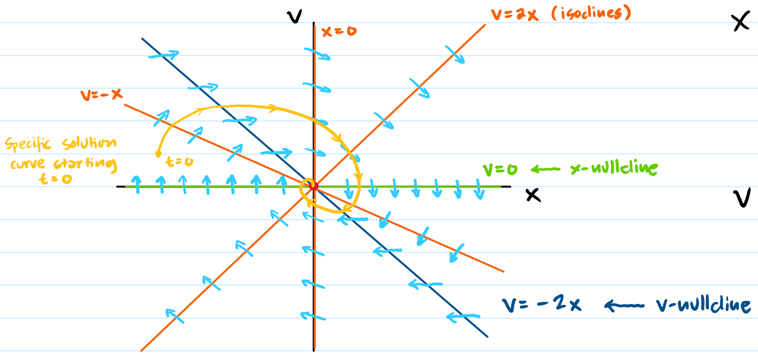
$$\lim_{m \to -b} \frac{-k}{m+b} x = \text{DNE}$$ ⟵ the x-nullcline is vertical

→ Isoclines: $v = \frac{-k}{m+b} x$

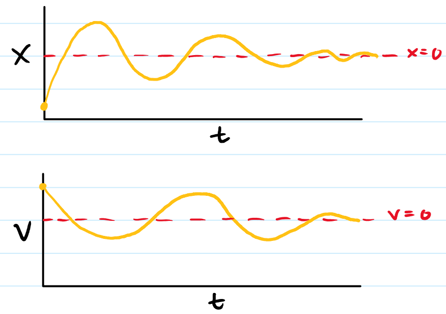
- Try $m = -2$; $v = \frac{-k}{-2+b} x$
- Try $m = -1$; $v = \frac{-k}{-1+b} x$
- Try $m = 0$; $v = \frac{-k}{b} x$ ⟵ v-nullcline
- Try $m = +1$; $v = \frac{-k}{1+b} x$
- Try $m = +2$; $v = \frac{-k}{2+b} x$

Vector Fields and Phase Portraits

* Example 1: $b = 1$, $k = 2$; $x' = v$
$v' = -2x - v$



Approximate solution trajectories on the t-x & t-v plane.



* the solution is oscillating with decreasing amplitude.

→ Nullclines: $v = 0$ → $x' = 0$, $v' = -2x$ } If $x < 0$, then $v' > 0$. (direction ↑) If $x > 0$, then $v' < 0$. (direction ↓)

$v = -2x$ → $x' = v$, $v' = 0$ } If $v < 0$, then $x' < 0$. (direction ←) If $v > 0$, then $x' > 0$. (direction →)

→ Isoclines: $m = -2$; $v = 2x$ → $x' = 2x$, $v' = -2x - 2x = -4x$ } If $x < 0$; then $x' < 0$ & $v' > 0$. (direction ↖) If $x > 0$; then $x' > 0$ & $v' < 0$. (direction ↘)

$m = -1$; $x = 0$ → $x' = v$, $v' = -v$ } If $v < 0$; then $x' < 0$ & $v' > 0$. ↖ If $v > 0$; then $x' > 0$ & $v' < 0$. ↘

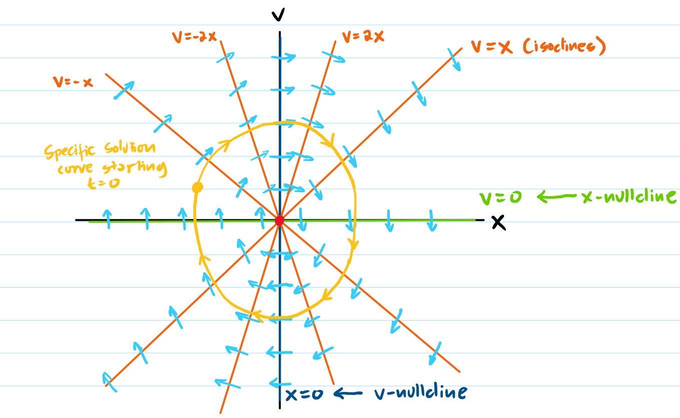
$m = 1$; $v = -x$ → $x' = -x$, $v' = -2x - (-x) = -x$ } If $x < 0$, then $x' > 0$ & $v' > 0$. ↗ If $x > 0$, then $x' < 0$ & $v' < 0$. ↙

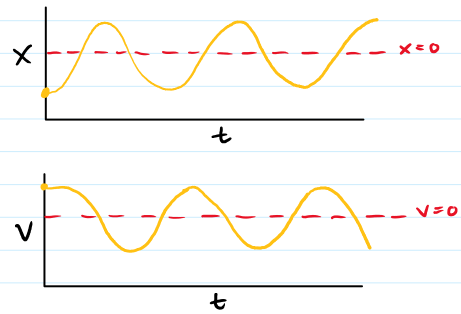
* Example 2: $b = 0$; $k = 2$; $x' = v$
$v' = -2x$

Approximate solution trajectories on the t-x & t-v plane.

✱ Example 2 : $b = 0$; $k = 2$; $x' = v$, $v' = -2x$

Approximate solution trajectories on the t-x & t-v plane.





✱ the solution is oscillating and never reaches equilibrium.

→ Nullclines : $v = 0$ ⟨ $x' = 0$, $v' = -2x$ ⟩ If $x < 0$, then $v' > 0$. ↑ (direction) ; If $x > 0$, then $v' < 0$. ↓

$x = 0$ ⟨ $x' = v$, $v' = 0$ ⟩ If $v < 0$, then $x' < 0$. ← (direction) ; If $v > 0$, then $x' > 0$. →

→ Isoclines : $m = -2$; $v = x$ ⟨ $x' = x$, $v' = -2x$ ⟩ If $x < 0$; then $x' < 0$ & $v' > 0$. ↖ (direction) ; If $x > 0$; then $x' > 0$ & $v' < 0$. ↘

$m = -1$; $v = 2x$ ⟨ $x' = 2x$, $v' = -2x$ ⟩ If $x < 0$; then $x' < 0$ & $v' > 0$. ↖ ; If $x > 0$; then $x' > 0$ & $v' < 0$. ↘

$m = 1$; $v = -2x$ ⟨ $x' = -2x$, $v' = -2x$ ⟩ If $x < 0$, then $x' > 0$ & $v' > 0$. ↗ ; If $x > 0$, then $x' < 0$ & $v' < 0$. ↙

$m = 2$; $v = -x$ ⟨ $x' = -x$, $v' = -2x$ ⟩ If $x < 0$, then $x' > 0$ & $v' > 0$ ↗ ; If $x > 0$, then $x' < 0$ & $v' < 0$ ↙

Phase Plane

A two-dimensional plane representing the two dependent variable of the linear system of ODEs.

Vector Field (direction field)

A field of direction vectors on a phase plane, where each vector corresponds to a slope and magnitude given by the linear system of ODEs.

Phase Portrait

A combination of a phase plane and a vector field with specific solution trajectories on the same plane.

Existence and Uniqueness for a system of 1st-order ODEs

The theory is similar as for the 1st-order single ODE.

- Examples

$\longrightarrow$ $$x' = \underbrace{v}_{f_1} \qquad v' = \underbrace{-kx - bv}_{f_2}$$

with initial conditions for any arbitrary constants x_0 & v_0

$x(0) = x_0$
$v(0) = v_0$

- Existence: continuous on $-\infty < x < \infty$ ✓, $-\infty < v < \infty$ ✓, $-\infty < t < \infty$ ✓ for both f_1 & f_2.
  Thus, solutions exists within the domain.

- Uniqueness:
  for f_1; $\frac{\partial f_1}{\partial x} = 0$, continuous on $-\infty < x < \infty$, $-\infty < v < \infty$ ✓ & $-\infty < t < \infty$ ✓
  $\frac{\partial f_1}{\partial v} = 1$, continuous on $-\infty < x < \infty$, $-\infty < v < \infty$ ✓ & $-\infty < t < \infty$ ✓
  Also, the initial conditions are within the continuous domain. ✓

  for f_2; $\frac{\partial f_2}{\partial x} = -k$, cont. on $x \in (-\infty, \infty)$, $v \in (-\infty, \infty)$ ✓ & $t \in (-\infty, \infty)$ ✓
  $\frac{\partial f_2}{\partial v} = -b$, cont. on $x \in (-\infty, \infty)$, $v \in (-\infty, \infty)$ ✓ & $t \in (-\infty, \infty)$ ✓
  Also, the initial conditions are within the cont. domain. ✓
  Thus, solutions are unique for any x_0 & v_0.

$\longrightarrow$ $$X' = \underbrace{-(\tfrac{1}{2})Y}_{f_1} \qquad Y' = \underbrace{(\tfrac{1}{2})X - (\tfrac{1}{3})Y}_{f_2}$$

with initial conditions for any constants X_0 & Y_0

$X(0) = X_0$
$Y(0) = Y_0$

- Existence: Cont. on $X \in (-\infty, \infty)$ ✓, $Y \in (-\infty, \infty)$ ✓, $t \in (-\infty, \infty)$ ✓ for both f_1 & f_2
  Thus, solutions exists within the domain.

- Uniqueness:
  for f_1; $\frac{\partial f_1}{\partial X} = 0$, $\frac{\partial f_1}{\partial Y} = -\frac{1}{2}$
  for f_2; $\frac{\partial f_2}{\partial X} = \frac{1}{2}$, $\frac{\partial f_3}{\partial Y} = -\frac{1}{3}$

  All constants.
  So, continuous for $X \in (-\infty, \infty)$ ✓, $Y \in (-\infty, \infty)$ ✓, $t \in (-\infty, \infty)$ ✓

  Also, initial conditions are within the continuous domain.
  Thus, solutions are unique for any X_0 & Y_0.