Existence and Uniqueness & Principle of Superposition

Objectives:

1. Introduce the Superposition principle
2. Existence theorem of 2nd-order ODEs
3. Uniqueness theorem of 2nd-order ODEs

Recall: Proposed solution Verification

* Example: $y'' - 3y = 0 \longrightarrow$ proposed solutions $y_1 = e^{\sqrt{3}t}$, $y_2 = e^{-\sqrt{3}t}$

• Verify
$$y_1 = e^{\sqrt{3}t}$$
$$y_1' = \sqrt{3}\, e^{\sqrt{3}t}$$
$$y_1'' = 3e^{\sqrt{3}t}$$

$$y'' - 3y = 0 \longrightarrow \cancel{3e^{\sqrt{3}t}} - \cancel{3e^{\sqrt{3}t}} = 0$$

$0 = 0$, so, y_1 is a solution.

• Verify
$$y_2 = e^{-\sqrt{3}t}$$
$$y_2' = -\sqrt{3}\, e^{-\sqrt{3}t}$$
$$y_2'' = 3e^{-\sqrt{3}t}$$

$$y'' - 3y = 0 \longrightarrow 3e^{-\sqrt{3}t} - 3e^{-\sqrt{3}t} = 0$$

Linear 2nd-Order ODEs

General form: $y'' + py' + qy = f(t)$ where p, q, and $f(t)$ are functions of t.

Homogeneous vs Non-homogeneous

- If $f(t) = 0$, then the ODE is homogeneous
- If $f(t) \neq 0$, then the ODE is non-homogeneous

Homogeneous 2nd-order ODEs

* Example: $\underset{p=0}{y''} - \underset{q=-1}{y} = \underset{f(t)=0}{0}$ with solutions $\underbrace{y_1 = e^{t} \text{ \& } y_2 = e^{-t}}_{\text{independent solutions}}$

General Solution: $y = c_1 y_1 + c_2 y_2$ where c_1 & c_2 are constants.

$$\downarrow$$

$y = c_1 e^{t} + c_2 e^{-t}$ ⟵ two constants because 2nd-order ODE

$$y = \underbrace{c_1 e^{t} + c_2 e^{-t}}_{\text{linear combination of } y_1 \text{ \& } y_2}$$

← two constants because 2nd-order ODE is integrated twice.

- Verify

$$y = c_1 e^{t} + c_2 e^{-t}$$
$$y' = c_1 e^{t} - c_2 e^{-t}$$
$$y'' = c_1 e^{t} + c_2 e^{-t}$$

$$y'' - y = 0 \longrightarrow (c_1 e^{t} + c_2 e^{-t}) - (c_1 e^{t} + c_2 e^{-t}) = 0$$
$$\cancel{c_1 e^{t}} - \cancel{c_1 e^{t}} + \cancel{c_2 e^{-t}} - \cancel{c_2 e^{-t}} = 0$$
$$0 = 0,$$ so, y is a solution.

The Superposition Principle

It states that for a linear 2nd-order homogeneous ODE,
if y_1 & y_2 are independent solutions, then
any <u>linear combination</u>

$$y = c_1 y_1 + c_2 y_2$$

where c_1 & c_2 are constants, is also a solution.

* Example: $y'' - y = 0 \longrightarrow$ general solution is $y = c_1 e^{t} + c_2 e^{-t}$

Given initial conditions: $y(0) = 1$ & $y'(0) = 0$, find c_1 & c_2.

Compute derivatives of y

$$y(t) = c_1 e^{t} + c_2 e^{-t}$$
$$y'(t) = c_1 e^{t} - c_2 e^{-t}$$

apply conditions

$$y(0) = c_1 \overset{1}{\cancel{e^{(0)}}} + c_2 \overset{1}{\cancel{e^{-(0)}}} = 1$$
$$y'(0) = c_1 \overset{1}{\cancel{e^{(0)}}} - c_2 \overset{1}{\cancel{e^{-(0)}}} = 0$$

Set-up linear equation

two equations two unknowns
$$\begin{cases} c_1 + c_2 = 1 \\ c_1 - c_2 = 0 \end{cases}$$

or

two lines intersecting

Solve for c_1 & c_2

$$c_1 = c_2 \longrightarrow c_1 = 1/2$$
$$\hookrightarrow c_2 + c_2 = 1$$
$$2c_2 = 1$$

$$C_2 = 1/2$$

So, $C_1 = 1/2$ & $C_2 = 1/2$

thus, $y = \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}$ ← specific solution

Non-Homogeneous 2nd-Order ODEs

* Example: $y'' - y = 1$
  $f(t) = 1 \neq 0$

We showed that $y = c_1 e^{t} + c_2 e^{-t}$
is the general solution due to the superposition principle.

Now, we propose another independent solution $y_3 = -1$.

- Verify $y_3 = -1$
  $y_3' = 0$
  $y_3'' = 0$

$$y'' - y = 0 \longrightarrow 0 - (-1) = 1$$
$1 = 1$, so, y_3 is a solution.

Thus, the general solution can be

$$y = C_1 e^{t} + C_2 e^{-t} - 1$$

due to the superposition principle.

General Solution of Non-homogeneous 2nd-order ODEs

Types of Solutions:

- A homogeneous Solution y_h is a solution to the associated homogeneous equation given $f(t) = 0$.
  this is also called the complementary equation
- A particular Solution y_p is one specific solution to the nonhomogeneous equation given $f(t) \neq 0$.

Superposition Principle: If y_h is the homogeneous solution and y_p is the particular solution, then

$$y = y_h + y_p$$

is also a solution.

* Example: $y'' - 2y = 3$

Let $y_h = C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t}$

and

$y_p = -3/2$

So, the general solution is

$$y = y_h + y_p$$

$$y = C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t} - 3/2 .$$

<u>Existence and Uniqueness Theorem</u>

Let p, q, and $f(t)$ be continuous on $[a,b]$, then the ODE

$$y'' + py' + qy = f(t) \qquad y(t_0) = y_0, \ y'(t_0) = y_0'$$

has a unique solution defined for all t in $[a,b]$.

* Example: $y'' + \frac{3t}{t^2-1} y' + \frac{\cos(t)}{t^2-1} y = e^t$, $y(0) = 4$, $y'(0) = 5$

$p = \frac{3t}{t^2-1}$ $\quad$ $q = \frac{\cos(t)}{t^2-1}$ $\quad$ $f(t) = e^t$

- $f(t) = e^t$ is continuous on $t \in (-\infty, \infty)$
- $q = \frac{\cos(t)}{t^2-1}$ is discontinuous at $t = \pm 1$ and continuous on $t \in (-\infty,-1) \cup (-1,1) \cup (1,\infty)$.
- $p = \frac{3t}{t^2-1}$ has the same discontinuities as q.

So, the ODE has a unique solution on the interval

$$t \in (-\infty,-1) \cup (-1,1) \cup (1,\infty).$$

No guaranteed solution at $t = \pm 1$.