Existence and Uniqueness & Slope Fields and Phase Diagrams

Objectives:

1. Existence and Uniqueness of solutions
2. Phase diagrams
3. Slope fields

Exponential growth/decay model : Bacteria population

- Basic Assumptions: 1. Unlimited resources for growth.
  2. They replicate by binary fission, meaning growth is proportional to the number present.

- The ODE model as an Initial Value Problem (IVP):

$$\frac{dP}{dt} = KP \;;\; P(0) = P_0$$

P_0 → initial population

P → # of bacteria

$\frac{dP}{dt}$ → change of the number of bacteria

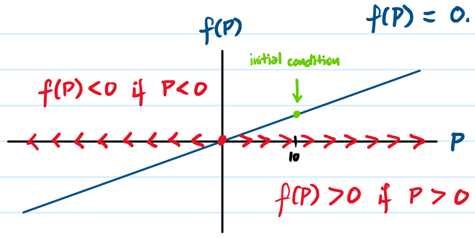
K → proportional constant

*Classification of the ODE

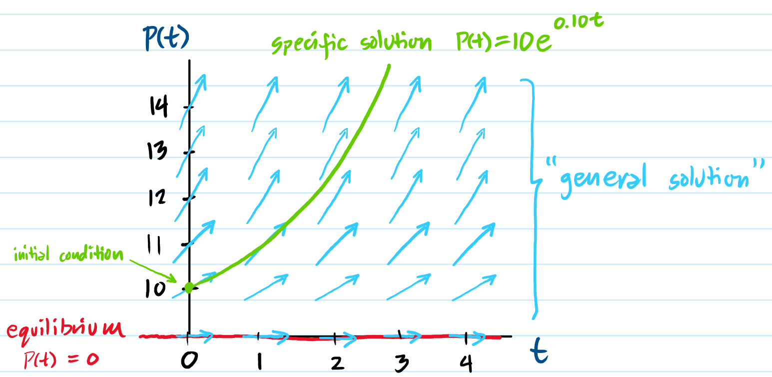
1. 1st-order
2. autonomous
3. homogeneous
4. linear

✱ Phase Diagram of $\frac{dP}{dt} = \boxed{0.10P}$ with condition $P(0)=10$

$f(P) = 0.10P$



✱ Slope Field of $\frac{dP}{dt} = 0.10P$ with condition $P(0)=10$



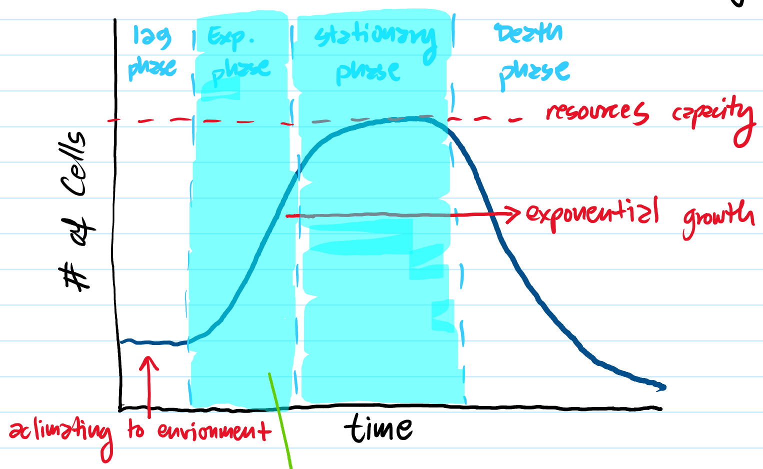
- General Solution in closed-form, meaning the solution is solved analytically:

$$P(t) = P_0 e^{kt}$$ where P_0 is the population at $t=0$.

Logistic Growth model : Bacteria population with limiting capacity

| lag | Exp. | stationary | Death

Logistic Growth model : Bacteria population with limiting capacity



This is what we modeled before using an ODE
Exponential growth equation $\frac{dB}{dt} = KB$

→ Modeling growth with limited capacity

Add an assumption : the environment on which the bacteria is growing with resources capacity in space and resources.

* the new ODE model *

$$\frac{dP}{dt} = KP - K\frac{P^2}{N}$$

→ resources capacity where limit is N. The bacteria starts to grow slowly as they compete with each other.

$\frac{dP}{dt}$: change in the number of Bacteria
KP: natural exponential growth

or

$$\frac{dP}{dt} = KP\left(1 - \frac{P}{N}\right)$$

→ Slope field

- Set ODE to $\frac{dP}{dt} = 0$ to find fixed points.

$$0 = KP\left(1 - \frac{P}{N}\right)$$

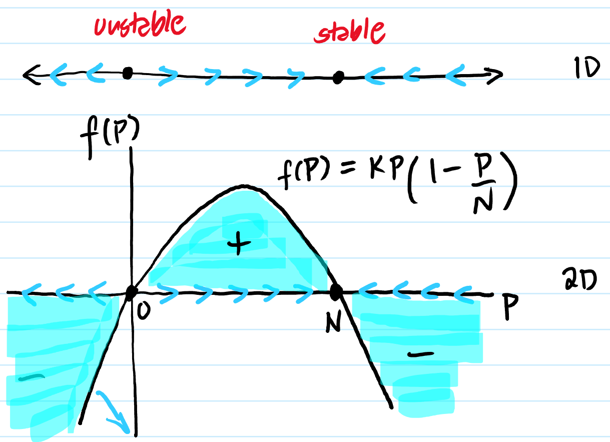
when $P = 0$ or
$\left(1 - \frac{P}{N}\right) = 0 \rightarrow P = N$

Fixed points → $P = 0$ and $P = N$

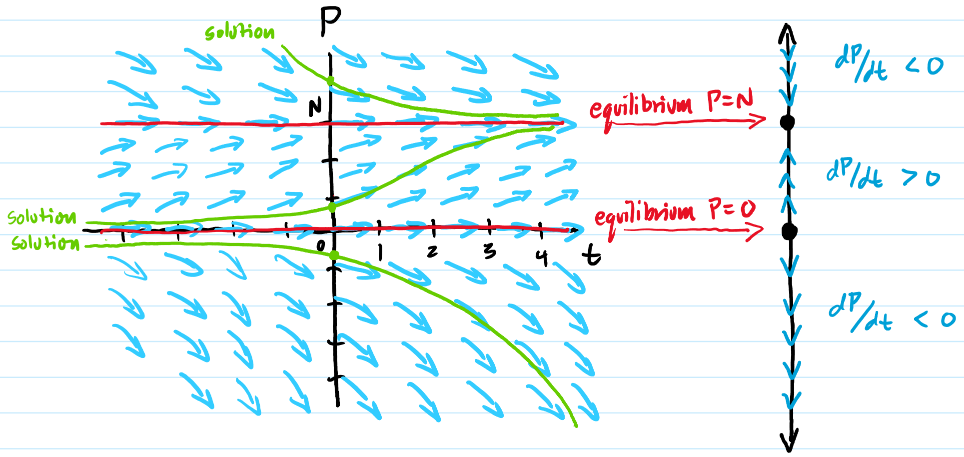
- $\frac{dP}{dt} > 0 \rightarrow KP\left(1 - \frac{P}{N}\right) > 0$ if $0 < P < N$

$$\frac{dP}{dt} < 0 \rightarrow KP\left(1-\frac{P}{N}\right) < 0 \text{ if } P<0 \text{ or } P>N$$

- Phase Diagram in 2D



- Slope Field



Slope Fields of ODE with Isoclines and Nullclines

- Example ODE: $\frac{dy}{dt} = t - y \xrightarrow{\text{rewrite}} \frac{dy}{dt} + y = t$

  * classifications
  - 1st order
  - Linear
  - Non-homo
  - non-auto

* Isoclines: a curve following vectors with the same slope.

$$\underset{\text{constant slope}}{c} = t - y \longrightarrow \overbrace{y = t + c}^{\text{Isocline equation}}$$

t: a line of 1 slope

$+c$: ↳ vertical shift

* Nullcline: a curve following vectors with zero slope. (critical points)

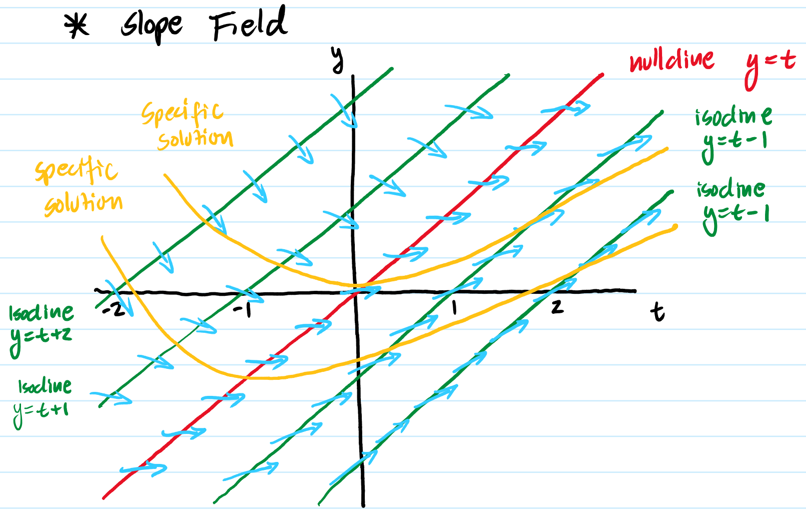
nullcline equation

$0 = t - y \longrightarrow \overbrace{y = t}\ + 0$

zero slope

↳ when $c = 0$.

a line of 1 slope



→ Slopes on the isoclines

- $y = t + 2 \to -2 = t - y \to \frac{dy}{dt} < 0$
- $y = t + 1 \to -1 = t - y \to \frac{dy}{dt} < 0$
- $y = t - 1 \to 1 = t - y \to \frac{dy}{dt} > 0$
- $y = t - 2 \to 2 = t - y \to \frac{dy}{dt} > 0$

Notations: 1. Let $\frac{dy}{dt} = f(y,t)$ be a 1st-order non-autonomous ODE.

2. Let $\frac{dy}{dt} = f(y)$ be a 1st-order autonomous ODE.

Existence and Uniqueness of Solutions

Suppose we are given an ODE.

Main Questions: 1. Does any solutions exist?
2. If any solutions exist, is it a unique solution?

* Sometimes solutions exists but we can't find them analytically at least *
* Sometimes initial value problems have non-unique solutions.

Examples:

- Given $\frac{dy}{dt} = t - y + 3$ with $y(0) = 1$.

* Does solution exists?

→ Check for continuity on the domain $-\infty < t < \infty$ and $\infty < y < \infty$.

Let $f(y,t) = \underbrace{t - y + 3}$.

continuous on the domain $-\infty < t < \infty$ and $-\infty < y < \infty$. ✓

the initial value of $(0,1)$ is in the interval. ✓

Thus, at least one solution exists in the domain.

* Is the solution unique?

→ check for continuity of $\underbrace{\frac{\partial}{\partial y} f(y,t)}$

partial derivative with respect to y of $f(y,t)$

partial derivative with respect to y of $f(y,t)$

$\frac{\partial}{\partial y} f(y,t) = -1 \rightarrow$ continuous on the domain $-\infty < t < \infty$ and $-\infty < y < \infty$. ✓

The initial value of $(0,1)$ is in the interval ✓

Thus, the solution is unique in the domain.

- Given $\frac{dy}{dt} = 5y^{4/5}$ with $y(0)=0$.

Existence: Let $f(y,t) = 5y^{4/5}$. $\rightarrow$ continuous on the domain $0 \le t < \infty$, $0 \le y < \infty$ ✓

the point $(0,0)$ is in the domain ✓

thus, a solution exists within the domain.

Uniqueness: $\frac{\partial}{\partial y} f(y,t) = \frac{\partial}{\partial y}\left(5y^{4/5}\right) = \frac{4}{y^{1/5}} \longrightarrow$ continuous on the domain $0 < t < \infty$, $0 < y < \infty$ ✓

but the point $(0,0)$ is not in the domain.

Thus, a unique solution is not guaranteed within the domain.

Existence and Uniqueness theorem for 1st-order ODE

Given an initial value problem $\frac{dy}{dt} = f(t,y)$ with $y(t_0) = y_0$.

Suppose that $\frac{dy}{dt} = f(t,y)$ and $\frac{\partial}{\partial y} f(t,y)$ are both continuous on the rectangular domain $R = \{(y,t) : |y-y_0| \le b, |t-t_0| \le a\}$, then there is a constant $0<\epsilon<a$ and a function $y(t)$ defined for t in the interval $(t_0-\epsilon, t_0+\epsilon)$ such that $y(t)$ is unique solution on this interval.