Integrating Factors

Objectives:

1. the method of integrating factors

Recall

- Product Rule of Derivatives

Suppose we have a product of functions $u(t)y(t)$.

$\longrightarrow \frac{d}{dt}u(t)y(t) = [uy]' \longrightarrow \frac{d}{dt}(u(t)y(t)) = u(t)\frac{dy}{dt} + \frac{du}{dt}y(t)$

$[uy]' = uy' + u'y$

Example: • $\sin(t)y(t) \longrightarrow \frac{d}{dt}(\sin(t)y(t)) = \sin(t)\frac{dy}{dt} + \cos(t)y(t)$

• In reverse: $t^2\frac{dy}{dt} + 2t\,y(t) \longrightarrow$ we know that $\frac{d}{dt}t^2 = 2t$

Let $u(t) = t^2$ & $du/dt = 2t$.

So, $u(t)\frac{dy}{dt} + \frac{du}{dt}y(t) = \frac{d}{dt}(u(t)y(t))$

The method of Integrating Factors (IF)

Suppose we have an ODE in 1st-order linear form,

$$\frac{dy}{dt} + py = q \longrightarrow \text{1st-order linear form}$$

where p and q are continuous functions of t.

General Solution using IF: $\frac{dy}{dt} + py = q$

- Set IF to $u = e^{\int p\,dt}$.
- multiply each side by u in ODE:

$$u\frac{dy}{dt} + upy = uq$$

- Let $\frac{du}{dt} = up$. Then, apply "reverse product" rule:

$$u\frac{dy}{dt} + \frac{du}{dt}y = uq$$

$$\frac{d}{dt}(uy) = uq$$

- Integrate both sides with respect to t:

$$\int \frac{d}{dt}(uy)\,dt = \int uq\,dt$$

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$$uy = \int uq\,dt$$

- Solve for y:

$$y(t) = \frac{1}{u}\int uq\,dt$$

Example:

- Find the general solution of following ODE,

$$\frac{dy}{dt} + \frac{3y}{t} = \frac{e^t}{t^3}$$

$\longrightarrow$ Form: $\dfrac{dy}{dt} + py = q$

$$p = \frac{3}{t} \qquad q = \frac{e^t}{t^3}$$

$\longrightarrow$ Find IF: Let $u = e^{\int p\,dt}$

$$u = e^{\int 3/t\,dt} \longrightarrow \int \frac{3}{t}\,dt = 3\ln(t)$$

$$u = e^{3\ln(t)}$$

$$u = t^3$$

$\longrightarrow$ Multiply u to ODE: $t^3\dfrac{dy}{dt} + t^3\dfrac{3}{t}y = t^3\dfrac{e^t}{t^3}$

$$t^3\frac{dy}{dt} + 3t^2 y = e^t$$

$\longrightarrow$ Integrate both sides: $\underbrace{\int\left(t^3\dfrac{dy}{dt} + 3t^2y\right)dt}_{\text{"reverse product" rule}} = \int e^t\,dt$

$$\int \frac{d}{dt}(t^3y)\,dt = \int e^t\,dt$$

$$t^3y = e^t + c$$

$$y(t) = \frac{1}{t^3}(e^t + c) \longrightarrow \text{general solution}$$

$\longrightarrow$ Verify: $\dfrac{dy}{dt} = \dfrac{1}{t^3}(e^t) - \dfrac{3}{t^4}(e^t + c)$

$$\frac{dy}{dt} = \frac{e^t}{t^3} - \frac{3e^t}{t^4} - \frac{3c}{t^4}$$

$$\downarrow$$

$$\frac{dy}{dt} + \frac{3y}{t} = \frac{e^t}{t^3}$$

$$\Downarrow$$

$$\frac{dy}{dt} + \frac{3y}{t} = \frac{e^t}{t^3}$$

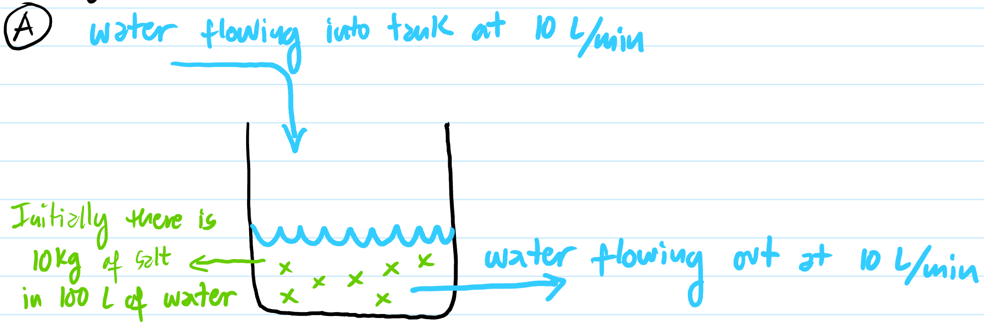
$$\frac{e^t}{t^3} - \cancel{\frac{3e^t}{t^4}} - \cancel{\frac{3c}{t^4}} + \cancel{\frac{3e^t}{t^4}} + \cancel{\frac{3c}{t^4}} = \frac{e^t}{t^3}$$

$$\frac{e^t}{t^3} = \frac{e^t}{t^3}$$

$$1 = 1 \checkmark$$

Mixing Problem

(A)



Basic Assumptions: 1. The water and salt are well mixed.

Main Question: How much salt will there be in the tank in 30 minutes?

* Model the rate of change of the salt content in the water. *

→ Let $S(t)$ be concentration of salt in the tank at time t in minutes, $S \geq 0$.
→ Let $\frac{dS}{dt}$ be the rate of change.

Main idea: $\underbrace{\text{rate of change} = \text{rate of flow in} - \text{rate of flow out}}_{\text{We need to account for the concentration of salt in the water}}$

- Water: * rate of flow in (water) = $10 L/min$.
  * rate of flow out (water) = $10 L/min$.
  Since in & out are equal, then the water inside the tank remains constant at 100 L.

- Salt: * concentration of salt = $\frac{S}{100} \frac{Kg}{L}$ (S → amount of salt; 100 → amount of water)

  * concentration of salt rate of flow in $= 0$ → because no salt is added

  * concentration of salt rate of flow out $= \left(\frac{S}{100} \frac{Kg}{\cancel{L}}\right)\left(10 \frac{\cancel{L}}{min}\right) = \frac{S}{10} \frac{Kg}{min}$

- ODE model: $\frac{dS}{dt} = 0 - \frac{S}{10}$, $S(0) = \frac{10}{100} \frac{Kg}{L} = \frac{1}{10} \frac{Kg}{L}$ → initial condition

* ODE model: $\frac{dS}{dt} = 0 - \frac{S}{10}$, $S(0) = \frac{10}{100}\frac{kg}{L} = \frac{1}{10}\frac{kg}{L}$ → initial condition

rate flow in

rate flow out

Rate of change of salt concentration

Rewrite : $\frac{dS}{dt} = -\frac{S}{10}$ → 1st-order linear autonomous homogeneous → seperable

* We can use separation of variables to solve the ODE *

$$\longrightarrow \frac{dS}{S} = -\frac{1}{10}\,dt$$

$$\int \frac{dS}{S} = -\int \frac{1}{10}\,dt$$

$$e^{\ln(S)} = e^{-\frac{t}{10} + C}$$

$$S = e^{-t/10 + c}$$

$$S = e^{-t/10} e^{c} \to C$$

$$S(t) = C e^{-t/10} \text{ kg/L} \to \text{general solution}$$

$$\longrightarrow \text{Since } S(0) = 1/10 \text{ kg/L}$$

$$1/10 = C_1 \cancel{e^{-0/10}}\ 1$$

$$C_1 = 1/10$$

$$S(t) = (1/10)\, e^{-t/10} \text{ kg/L} \to \text{particular solution}$$

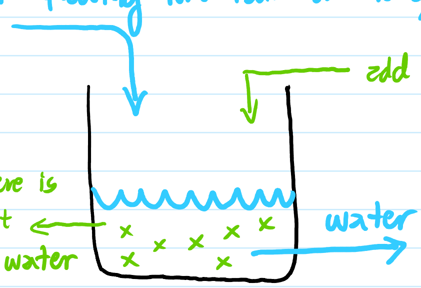
* at $t = 30$ min, $S(30) = (1/10)\, e^{-30/10} \approx 0.00498$ kg/L of salt

Ⓑ

water flowing into tank at 10 L/min

add salt 1/10 kg/min

salt concentration $\frac{1/10}{10}\frac{kg/min}{L/min} = \left(\frac{1}{10}\frac{kg}{\cancel{min}}\right)\left(\frac{1}{10}\frac{\cancel{min}}{L}\right) = \frac{1}{100}\frac{kg}{L}$



Initially there is 10 kg of salt in 100 L of water

water flowing out at 10 L/min

* How much salt is in the tank in 30 min.? *

Use same basic assumptions as Ⓐ.

* modified ODE model *

$$\frac{dS}{dt} = \frac{1}{100} - \frac{S}{10}, \quad S(0) = 1/10 \ \text{Kg/L} \longrightarrow \text{initial condition}$$

$\frac{dS}{dt}$ → Rate of change of salt concentration

$\frac{1}{100}$ → rate flow in

$\frac{S}{10}$ → rate flow out

Rewrite: $$\frac{dS}{dt} = \frac{1}{10}\left(\frac{1}{10} - S\right) \longrightarrow \text{1st-order linear autonomous homogeneous} \longrightarrow \text{seperable}$$

* Use separation of variables *

$$\longrightarrow \frac{dS}{\left(\frac{1}{10} - S\right)} = \frac{1}{10}\,dt$$

$$\int \frac{1}{(1/10 - S)}\,dS = \int \frac{1}{10}\,dt$$

$$-\ln(1/10 - S) = t/10 + C$$

$$e^{\ln(1/10 - S)} = e^{-t/10 - C} \longrightarrow \text{absorb to } C$$

$$1/10 - S = e^{-t/10 + C}$$

$$1/10 - S = e^{-t/10} e^{C} \longrightarrow \text{absorb to } C$$

$$-S = Ce^{-t/10} - 1/10$$

$$S(t) = 1/10 - Ce^{-t/10} \ \text{Kg/L} \longrightarrow \text{general solution}$$

$\longrightarrow$ Since $S(0) = 1/10$ Kg/L, then

$$S(0) = 1/10 - Ce^{-0/10}$$

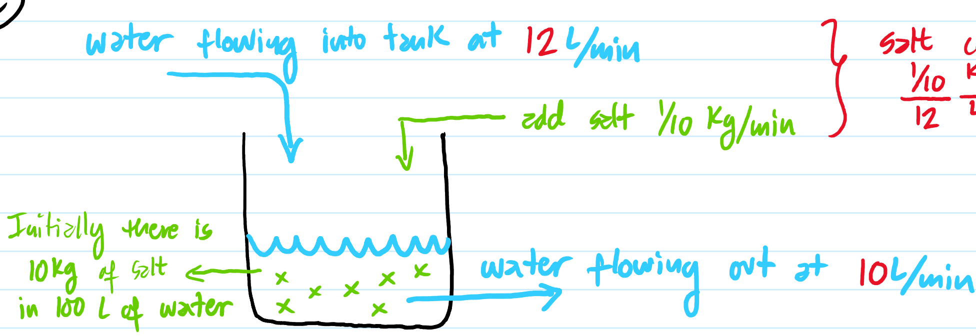
$$1/10 = 1/10 - C$$

$$C = 0$$

$$S(t) = 1/10 \longrightarrow \text{particular solution / equilibrium}$$

* at $t = 30$ min, $S(30) = 1/10 \approx 0.10$ Kg/L of salt

(c)



salt concentration

$$\frac{1/10 \ \text{Kg/min}}{12 \ \text{L/min}} = \left(\frac{1}{10}\frac{\text{Kg}}{\cancel{\text{min}}}\right)\left(\frac{1}{12}\frac{\cancel{\text{min}}}{\text{L}}\right) = \frac{1}{120}\frac{\text{Kg}}{\text{L}}$$

→ the in flow is greater than the out flow, which means that the volume of water is not constant.

* what is the amount of water at time t.

→ Initial conditions: $S(0) = 1/10$ Kg/L in a 100 L of water

$$V = \underbrace{V_0}_{\text{initial volume}} + \underbrace{(V_{in} - V_{out})t}_{\text{Change in volume at time } t.}$$

$$V(t) = V_0 + (V_{in} - V_{out})t$$

$$V(t) = 100 + 2t$$ → amount of water at time t.

* modified ODE model *

→ concentration of salt: $\dfrac{S}{V} = \dfrac{S}{100+2t}$

→ $\dfrac{dS}{dt} = \dfrac{1}{120} - \left(\dfrac{S}{100+2t}\right)$, $S(0) = 1/10$ Kg/L → 1st order linear homogeneous non-autonomous → not seperable

$\dfrac{dS}{dt}$ ↓ rate of change salt concentration

$\dfrac{1}{120}$ ↓ rate flow in

$\left(\dfrac{S}{100+2t}\right)$ ↓ rate flow out

* How do we solve a non-separable linear ODE? *
Use the Integrating Factors (IF) method.

$$\underbrace{\frac{dS}{dt}}_{S'} + \underbrace{\left(\frac{1}{100+2t}\right)}_{P(t)} S = \underbrace{\frac{1}{120}}_{Q(t)}$$ → 1st-order linear form

$$u(t)\frac{dS}{dt} + \underbrace{u(t)P(t)}_{\frac{du}{dt}} S = u(t)\frac{1}{120}$$ → multiply each side by an arbitrary function $u(t)$.

↳ let $\dfrac{du}{dt} = u(t)P(t)$

$$\frac{du}{dt} = u\left(\frac{1}{100+2t}\right)$$ → this is an ode that can be solved using SOV

$$\frac{du}{u} = \frac{dt}{100+2t}$$

$$\int \frac{du}{u} = \int \frac{dt}{100+2t}$$

$$\ln(u) = \frac{1}{2}\ln(100+2t)$$

$$u = \sqrt{100+2t}$$

$$\sqrt{100+2t}\,S' + \underbrace{\frac{\sqrt{100+2t}}{100+2t}\,S}_{} = \frac{\sqrt{100+2t}}{120}$$

Apply "reverse product" rule

$$\left(\sqrt{100+2t}\,S\right)' = \frac{\sqrt{100+2t}}{120}$$

$$\int\left(\sqrt{100+2t}\,S\right)' dt = \int \frac{\sqrt{100+2t}}{120}\,dt$$

$$\sqrt{100+2t}\,S = \frac{(100+2t)^{3/2}}{360} + C$$

$$S(t) = \frac{100+2t}{360} + \frac{C}{\sqrt{100+2t}} \longrightarrow \text{general solution}$$

Since $S(0) = 1/10$ kg/L, then

$$S(0) = \frac{100}{360} + \frac{C}{\sqrt{100}}$$

$$\frac{1}{10} = \frac{100}{360} + \frac{C}{10}$$

$$\frac{1}{10} = \frac{5}{18} + \frac{C}{10}$$

$$C = -\frac{16}{9}$$

$$S(t) = \frac{100+2t}{360} - \frac{16}{9\sqrt{100+2t}} \longrightarrow \text{specific solution}$$

$*$ @ $t=30$ min, $S(30) = \dfrac{100+2(30)}{360} + \dfrac{16}{9\sqrt{100+2(30)}} \approx 0.3039$ kg/L of salt