Method of Reduction of Order

Objectives:

1. Introduce the method of reduction of order
2. Show how the general solution with repeated root case has the 2nd solution with t multiple.
3. Introduce a different trial solution.

Recall: The characteristic equation

Given a linear 2nd-order ODE with constant coefficients

$$x'' + bx' + kx = 0 \longleftarrow \text{homogeneous}$$

$$\Big\downarrow \text{ let } y = e^{rt}$$

$$r^2 + br + k = 0 \longleftarrow \text{characteristic equation}$$

- To determine the general solution
  - If r is distinct real roots r_1 & r_2, then

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

  - If r is a repeated real root $r = r_1 = r_2$, then

$$x = c_1 e^{rt} + c_2 t e^{rt}$$

  - If r is a complex conjugate root $r = \alpha + \beta i$

$$x = e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right)$$

The Repeated Root Case

Example: $x'' - 4x' + 4x = 0$

$$\downarrow$$

$$r^2 - 4r + 4 = 0$$
$$(r-2)^2 = 0$$
$$r = r_1 = r_2 = 2.$$

So, $x_1 = e^{2t}$.

We want to find a 2nd independent solution x_2.

Let $x_2 = v x_1$ for some v function of t.

$$\downarrow$$
$$x_2 = v e^{2t}$$
$$x_2' = v' e^{2t} + 2v e^{2t}$$
$$x_2'' = v'' e^{2t} + 2v' e^{2t} + 2v' e^{2t} + 4v e^{2t}$$
$$= v'' e^{2t} + 4v' e^{2t} + 4v e^{2t}$$

$$x'' - 4x' + 4x = 0$$
$$(v'' e^{2t} + 4v' e^{2t} + 4v e^{2t}) - 4(v' e^{2t} + 2v e^{2t}) + 4v e^{2t} = 0$$
$$e^{2t}(v'' \cancel{+4v'} \cancel{+4v} \cancel{-4v'} \cancel{-8v} \cancel{+4v}) = 0$$
$$\downarrow$$
$$e^{2t} v'' = 0 \; , \; e^{2t} \neq 0$$
$$\downarrow$$
$$v'' = 0$$
$$\int v'' = \int 0$$
$$\int v' = \int c$$
$$v = ct + K$$

So, $x_2 = (ct + K)e^{2t}$

Thus, $x = C_1 x_1 + C_2 x_2$

$$= C_1 e^{2t} + C_2 (ct + K) e^{2t}$$
$$= C_1 e^{2t} + C_2 c t e^{2t} + C_2 K e^{2t}$$
$$= \underbrace{(C_1 + C_2 K)}_{C_1} e^{2t} + \underbrace{C_2 c}_{C_2} t e^{2t}$$

$x = C_1 e^{2t} + C_2 t e^{2t}$ ⟵ general solution

$$\underbrace{\quad}_{C_1} \qquad \underbrace{\quad}_{C_2}$$

$$x = C_1 e^{2t} + C_2 t e^{2t}$$ ← general solution

Another Class of Linear 2nd-order ODE

Example: $2t^2 y'' + ty' - 3y = 0, \quad t > 0$ ← 2nd-order, linear, non-auto, homo (non-constant coefficients)

Given $y_1 = t^{-1}$ as a solution.

* Method of Reduction of Order

Let $y_2 = v y_1$ for some function v. ← Similar method to what we did in the previous example

$$\downarrow$$

$$y_2 = v t^{-1}$$

$$y_2' = v' t^{-1} - v t^{-2}$$

$$y_2'' = v'' t^{-1} + v'(-t^{-2}) - v' t^{-2} + 2 v t^{-3}$$

$$= v'' t^{-1} - 2v' t^{-2} + 2 v t^{-3}$$

$$2t^2 y'' + ty' - 3y = 0$$

$$\downarrow$$

$$2t^2 (v'' t^{-1} - 2v' t^{-2} + 2v t^{-3}) + t(v' t^{-1} - v t^{-2}) - 3 v t^{-1} = 0$$

$$2tv'' - 4v' + 4vt^{-1} + v' - vt^{-1} - 3vt^{-1} = 0$$

$$2tv'' (-4+1)v' + (\cancel{4v - v - 3v}) t^{-1} = 0$$

$$2tv'' - 3v' = 0$$

$$\downarrow$$ Let $w = v'$. (substitution method) $\Rightarrow w' = v''$

$$2tw' - 3w = 0$$

$$2tw' = 3w$$

$$w' = \frac{3w}{2t}$$ ← 1st-order, linear, non-auto, homo (Separable ODE)

$$\int \frac{dw}{w} = \int \frac{3}{2t} dt$$

$$\ln(w) = \frac{3}{2} \ln(t) + c$$

$$e^{\ln(w)} = e^{\ln(t^{3/2}) + c}$$

$$w = c\, t^{3/2}$$

$e^{\ln(\ldots)} = e^{\ln(t^{\ldots})}$

$$w = c_1 t^{3/2}$$

Since $w = v'$, then $v' = c_1 t^{3/2}$

$\downarrow$ direct integration

$$\int v' = \int c_1 t^{3/2}$$

$$v = \frac{c_1 t^{5/2}}{5/2} + k$$

$$v = \frac{2}{5} c_1 t^{5/2} + k$$ ⟵ choose $c_1 = \frac{5}{2}$ & $k = 0$ to clear out fractions.

$$v = t^{5/2}$$

So, $y_2 = v y_1$

$$y_2 = t^{5/2} t^{-1}$$

$$y_2 = t^{3/2}$$

thus, $y = c_1 y_1 + c_2 y_2$

$$y = c_1 t^{-1} + c_2 t^{3/2}$$ ⟵ general solution

The Euler-Cauchy Equations (or Cauchy-Euler Equations)

A linear 2nd-order ODE of the form

$$a t^2 y'' + b t y' + c y = 0$$ for some constant $a, b, c \in \mathbb{R}$.

* A different Ansatz: Let $y = t^r$ for some constant r.

$$\downarrow$$

$$y = t^r$$
$$y' = r t^{r-1}$$
$$y'' = r(r-1) t^{r-2}$$

$$at^2y'' + bty' + cy = 0$$

$$\downarrow$$

$$at^2(r(r-1)t^{r-2}) + bt(rt^{r-1}) + ct^r = 0$$

$$ar(r-1)t^r + brt^r + ct^r = 0$$

$$t^r \underbrace{(ar(r-1) + br + c)}_{\text{has to equal zero}} = 0 \quad , \quad t^r \neq 0$$

$$ar(r-1) + br + c = 0$$

$$ar^2 - ar + br + c = 0$$

$$ar^2 + (b-a)r + c = 0 \longleftarrow \text{characteristic equation}$$

$$\downarrow$$

the roots of this equation can be used to find the solution, just like the constant coefficient case.

$*$ General Solution:

- If r is distinct real roots r_1 & r_2, then

$$y = t^{r_1} + t^{r_2}$$

- If r is a repeated real root $r = r_1 = r_2$, then

$$y = (c_1 + c_2 \ln(t))\, t^r$$

- If r is a complex conjugate root $r = \alpha + \beta i$

$$y = t^{\alpha}\left(c_1 \cos(\beta \ln(t)) + c_2 \sin(\beta \ln(t))\right)$$