Separation of Variables

Objectives:

1. the method of separation of variables

Recall

- Chain Rule of Derivatives

Suppose we have a composite function $f(g(t))$.

$$\longrightarrow \frac{d}{dt} f(g(t)) = [f(g(t))]' \longrightarrow \frac{df}{dt} = \frac{df}{dg}\frac{dg}{dt}$$

$$[f(g(t))]' = f'(g(t))g'(t)$$

Example: • $\ln(P(t)) \longrightarrow \frac{d}{dt}\ln(P(t)) = \frac{1}{P(t)}\frac{dP}{dt}$

• In reverse: $2y^3\frac{dy}{dt} \longrightarrow 2y^3\frac{dy}{dt} = \frac{d}{dt}\frac{y^4}{2}$

$$\downarrow$$

$$\frac{4y^3}{2} = 2y^3\frac{dy}{dt} \checkmark$$

Separation of Variables (SOV)

* SOV only works on separable ODEs of the form

$$\frac{dy}{dt} = f(y)g(t).$$

Note that y is a function of t.

Examples:

$\longrightarrow$ Separable ODEs

- $\frac{dy}{dt} = y \rightarrow$ exponential growth model

  $\hookrightarrow f(y) = y$ & $g(t) = 1$

- $\frac{dy}{dt} = y(1-y) \rightarrow$ logistic growth model

  $\hookrightarrow f(y) = y(1-y)$ & $g(t) = 1$

- $\frac{dy}{dt} = \frac{t}{y}$

  $\hookrightarrow f(y) = 1/y$ & $g(t) = t$

→ Non Separable ODEs

- $\frac{dy}{dt} = 2y + t^2$ → addition between y & t variables
- $\frac{dy}{dt} = \sin(y) + ty$ → can not factor out the y variable
- $\frac{dy}{dt} = \cos(t)y + 1$ → the "+1" makes the term $\cos(t)y$ nonseparable

* the SOV method (in general)

$$\frac{dy}{dt} = f(y)g(t)$$ → Take ODE in separable form

$$\boxed{\frac{1}{f(y)}\frac{dy}{dt}} = g(t)$$ → Divide $f(y)$ in both sides

↓ apply chain rule in reverse

$$\frac{d}{dt}\ln(f(y)) = g(t) \quad \text{or} \quad [\ln(f(y))]' = g(t)$$

$$\int \frac{d}{dy}\ln(f(y))\,dt = \int g(t)\,dt$$ → Apply FTC

$$e^{\ln(f(y))} = e^{\int g(t)\,dt}$$

$$f^{-1}(f(y)) = f^{-1}\left(e^{\int g(t)dt}\right)$$

$$y(t) = f^{-1}\left(e^{\int g(t)dt}\right)$$

This part will only work if the ODE is linear and separable, meaning for ODEs of the form $\frac{dy}{dt} = y\,g(t)$.

Otherwise, you need to separate the variables and integrate directly for non-linear and separable ODEs.

don't forget the "+C" when computing the final integral.

Example:

→ $\frac{dy}{dt} = \frac{t}{y}$, $y(0) = 1$ IVP

$$\frac{dy}{dt} = \underbrace{\left(\frac{1}{y}\right)}_{f(y)}\underbrace{t}_{g(t)}$$

$$\left(\frac{y}{1}\right)\frac{dy}{dt} = \left(\frac{y}{1}\right)\left(\frac{1}{y}\right)t$$

$$\boxed{y\frac{dy}{dt}} = t$$

↓ apply chain rule in reverse

$$\frac{d}{dt}\left(\frac{y^2}{2}\right) = t \quad \text{or} \quad \left(\frac{y^2}{2}\right)' = t$$

$$\frac{d}{dy}\left(\frac{y^2}{2}\right) = t \quad \text{or} \quad \left(\frac{y^2}{2}\right)' = t$$

$$\int \frac{d}{dy}\left(\frac{y^2}{2}\right) dy = \int t \, dt$$

$$\frac{y^2}{2} = \frac{t^2}{2} + C$$ → integration constant

$$y^2 = t^2 + 2C$$

$$y = \sqrt{t^2 + 2C}$$

common practice of "absorbing" constants

$$\underbrace{y(t) = \pm\sqrt{t^2 + C_1}}_{\text{general solution}}$$

Since $y(0) = 1$, then $y(0) = \pm\sqrt{0^2 + C_1}$

$1 = \pm\sqrt{C_1}$

$C_1 = 1$.

So, $\underbrace{y(t) = \pm\sqrt{t^2+1}}_{\text{specific solution}}$

Verify: $\frac{dy}{dt} = \frac{1}{2}(t^2+1)^{-1/2}\, 2t$

$= \frac{t}{\pm\sqrt{t^2+1}} \longrightarrow \frac{dy}{dt} = \frac{t}{y}$

$$\frac{t}{\sqrt{t^2+1}} = \frac{t}{\pm\sqrt{t^2+1}}$$

$1 = 1$ ✓ verified

* SOV shortcut

$\frac{dy}{dt} = f(y)\,g(t)$ → take ODE in separable form

$\frac{dy}{\cancel{dt}}\cancel{dt} = f(y)\,g(t)\,dt$ → multiply "dt" on both sides

$\left(\frac{1}{f(y)}\right)\frac{dy}{dt} = \left(\frac{1}{\cancel{f(y)}}\right)\cancel{f(y)}\,g(t)\,dt$ → divide $f(y)$ on both sides

$\frac{1}{f(y)}\,dy = g(t)\,dt$

$\int \frac{1}{f(y)}\,dy = \int g(t)\,dt$ → integrate

$\downarrow$

Solve for $y(t)$.

Example: IVP

$\longrightarrow \frac{dB}{dt} = KB - K\frac{B^2}{N}$, $B \geq 0$, $B(0) = B_0$ $\longrightarrow$ 1st-order non-linear homogeneous autonomous $\longrightarrow$ separable

parameters $\begin{cases} \bullet\ K \text{ is the growth constant},\ K > 0 \\ \bullet\ N \text{ is the limiting capacity},\ N > 0 \\ \bullet\ B_0 \text{ is the initial value},\ B_0 > 0 \end{cases}$

$$\frac{dB}{dt} = KB\left(1 - \frac{B}{N}\right)$$

$$\frac{dB}{\cancel{dt}}\cancel{dt} = KB\left(1 - \frac{B}{N}\right)dt$$

$$\frac{dB}{KB\left(1 - \frac{B}{N}\right)} = \frac{\cancel{KB\left(1 - \frac{B}{N}\right)}\,dt}{\cancel{KB\left(1 - \frac{B}{N}\right)}}$$

$$\boxed{\int \frac{1}{KB\left(1 - \frac{B}{N}\right)}\,dB} = \int dt$$

partial fraction decomposition

$$\frac{1}{KB\left(1 - \frac{B}{N}\right)} = \frac{a}{KB} + \frac{b}{1 - B/N}$$

$$1 = a(1 - B/N) + bKB$$

$$1 = a - (a/N)B + bKB$$

$$1 = a + B(-a/N + bK) \longrightarrow B;\quad 0 = -a/N + bK \longrightarrow b = 1/KN$$

$$\text{const};\quad 1 = a \longrightarrow a = 1$$

$$\frac{1}{KB\left(1 - \frac{B}{N}\right)} = \frac{1}{KB} + \frac{1/KN}{1 - B/N}$$

$$\int \left(\frac{1}{KB} + \frac{1/KN}{1 - B/N}\right)dB = \int dt$$

$$\frac{1}{K}\int \frac{1}{B}\,dB + \frac{1}{KN}\boxed{\int \frac{1}{1 - B/N}\,dB} = \int dt$$

u-substitution

let $u = 1 - B/N$, $du = -1/N\,dB$

$$\int \frac{1}{1 - B/N}\,dB = \int \frac{1}{u}(-N)\,du$$

let $u = 1 - B/N$, $du = -1/N \, dB$

$$\int \frac{1}{1-B/N} dB = \int \frac{1}{u}(-N)\,du$$
$$= -N \int \frac{1}{u} du$$
$$= -N \ln(u)$$
$$= -N \ln\left(1 - \frac{B}{N}\right)$$

$$\frac{1}{k}\ln(B) + \frac{1}{k\cancel{N}}\left(-\cancel{N}\ln\left(1-\frac{B}{N}\right)\right) = t + C$$ → integration constant

$$\frac{1}{k}\ln(B) - \frac{1}{k}\ln\left(1-\frac{B}{N}\right) = t + C$$
$$\frac{1}{k}\boxed{\left(\ln(B) - \ln\left(1-\frac{B}{N}\right)\right)} = t + C$$

law of logs
$\ln(x) - \ln(y) = \ln(x/y)$

$$\frac{1}{k}\left(\ln\left(\frac{B}{1-B/N}\right)\right) = t + C$$
$$\ln\left(\frac{B}{1-B/N}\right) = kt + kC$$

e e

$$\frac{B}{1-B/N} = \boxed{e^{kt+kC}}$$

law of exponents
$e^{x+y} = e^x e^y$

$$\frac{B}{1-B/N} = e^{kt}e^{kC}$$
$$\frac{B}{\left(\frac{N-B}{N}\right)} = e^{kt}e^{kC}$$
$$\frac{BN}{N-B} = e^{kt}e^{kC}$$
$$BN = (N-B)e^{kt}e^{kC}$$
$$BN = Ne^{kt}e^{kC} - Be^{kt}e^{kC}$$
$$B(N + e^{kt}e^{kC}) = N e^{kt}e^{kC}$$
$$B(t) = \frac{Ne^{kt}e^{kC}}{N + e^{kt}e^{kC}}$$ absorb constant to C

$$B(t) = \frac{Ne^{kt}C}{N + Ce^{kt}}$$ → general solution

Since $B(0) = B_0$, $B(0) = \dfrac{N\cancel{e^{k(0)}}^{1}C}{N + C\cancel{e^{k(0)}}^{1}}$

$$B_0 = \frac{NC}{}$$

$$\overline{\quad N + C\cancel{e^{k(0)}}\ 1 \quad}$$

$$B_0 = \frac{NC}{N+C}$$

$$B_0(N+C) = NC$$

$$B_0N + B_0C = NC$$

$$B_0N = NC - B_0C$$

$$B_0N = C(N + B_0)$$

$$C = \frac{B_0N}{N - B_0}$$

So, $B(t) = \dfrac{N\left(\dfrac{B_0N}{N-B_0}\right)e^{kt}}{N + \left(\dfrac{B_0N}{N-B_0}\right)e^{kt}}$. $\longrightarrow$ specific solution

Equilibriums: $\longrightarrow \dfrac{dB}{dt} = 0$ if $B = 0$ or $B = N$.

$\longrightarrow \lim\limits_{t\to\infty} B(t) \overset{LH}{=} \lim\limits_{t\to\infty} \dfrac{\cancel{k}N\cancel{C}}{\cancel{k}\cancel{C}}$ $\longrightarrow$ L'hopital's rule

$\lim\limits_{t\to\infty} B(t) = N$ ✓

$\longrightarrow$ If $B_0 = 0$, then $C = 0$ and $B(t) = 0$ ✓

$\longrightarrow$ If $B_0 = N$, then $C =$ UND and $B(t) = N$ ✓