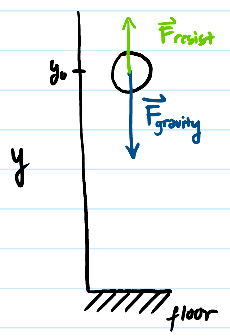
System of 1st-Order ODEs & Equilibriums

Objectives

1. Introduce system of 1st-order ODEs
2. Modeling physical systems using system of ODEs
3. Introducing Nullclines of 1st-order systems

Recall: Falling Object with Air Resistance



$\longrightarrow$ ODE: $\vec{F} = \vec{F}_{gravity} + \vec{F}_{resist}$

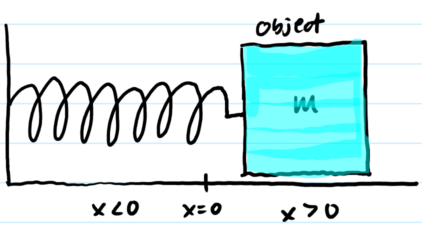
$$\downarrow$$

$$my'' = g - ky'$$

or

$$my'' + ky' = g \quad \longleftarrow \text{2nd-order, linear, auto, non-homo}$$

Recall: Horizontal Spring-Mass



$$\vec{F} = \vec{F}_{friction} + \vec{F}_{spring} + \vec{F}_{external}$$

$$\vec{F} - \vec{F}_{friction} - \vec{F}_{spring} = \vec{F}_{external}$$

$$\downarrow$$

$$mx'' - (-bx') - (-kx) = 0 \quad \longleftarrow \text{assuming no external forces}$$

$$\downarrow$$

$$mx'' + bx' + kx = 0$$

or

$$x'' + \frac{b}{m}x' + \frac{k}{m}x = 0$$

* To simplify the ODE, we let $m = 1$.

$$x'' + bx' + kx = 0 \quad \longleftarrow \text{2nd-order homogeneous linear ODE with constant coefficients}$$

Converting a Linear 2nd-order ODE in a 1st-order System

* $x'' + bx' + kx = 0 \quad \longleftarrow$ 2nd-order, linear, auto, homo constant coefficients

$$\downarrow$$

Let $v = x'$ and $v' = x''$

$$v' + bv + kx = 0$$

$$v' = -kx - bv$$

$$\downarrow$$

Velocity $\longrightarrow$ $x' = v$ $\Big\}$ 1st-order system of ODEs

acceleration $\longrightarrow$ $v' = -kx - bv$ with dependent variables x & v

Velocity ⟶ $x' = v$
acceleration ⟶ $v' = -kx - bv$
} 1st-order system of ODEs with dependent variables x & v.

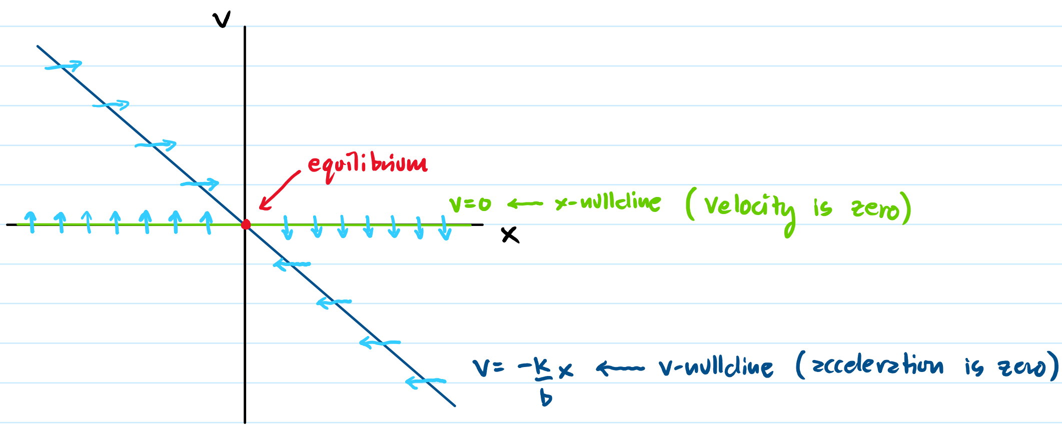
⟶ We know that $x(t)$ is the displacement.
So, x' is the velocity. So, if $v = x'$, then v is the velocity.
If $v' = x''$, then v' is the acceleration.

Nullclines

* Nullclines are when the derivatives are zero (critical points).

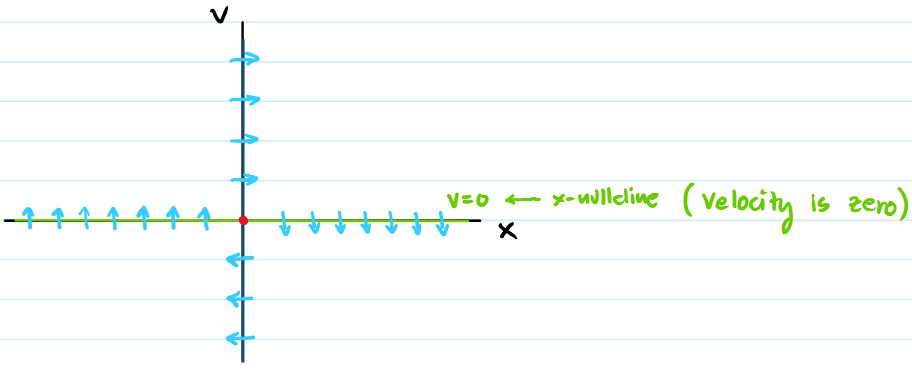
Example: $x' = v$
$v' = -kx - bv$

- x-nullcline: $x' = 0 \longrightarrow v = 0 \longrightarrow$ no change in x
  $v' = -kx$
  - If $x < 0$, then $v' > 0$. ↑
  - If $x > 0$, then $v' < 0$. ↓
- v-nullcline: $0 = -kx - bv$
  $bv = -kx$
  $v = -\frac{k}{b}x \longrightarrow$ no change in v
  $x' = v$
  - If $v < 0$, then $x' < 0$. ⟵
  - If $v > 0$, then $x' > 0$. ⟶
- Viewed on a x-v axis (plane).



- In this case the x-nullcline is always v=0.
  the v-nullcline depends on k & b parameters.
  ⟶ Let k be fixed. If $b \to 0$, then v-nullcline gets steeper.
  If $b \to \infty$, then v-nullcline gets horizontal.

If $b=0$ (no friction),



$x=0$ ← v-nullcline (acceleration is zero)

Equilibriums

* An equilibrium is a constant solution when $x'=0$ & $v'=0$.
  In other words, the x-nullcline and v-nullcline intersect.

Example: $x' = v$
$v' = -kx - bv$

$$\downarrow$$

$$\left.\begin{aligned} 0 &= v \\ 0 &= -kx - bv \end{aligned}\right\}$$ homogeneous linear system
two equations & two unknowns

$$\downarrow$$

$$\left.\begin{aligned} v &= 0 \\ x &= 0 \end{aligned}\right\}$$ equilibrium solution