Undetermined Coefficients for Nonhomogeneous Linear ODEs

Objectives:

1. Introduce the method of Undetermined Coefficients

Recall: <u>2nd-order linear ODE with constant coefficients</u>

$$y'' + py' + qy = f(t)$$

where p & q are constants, and $f(t)$ is a function.
If $f(t) = 0$, then it is homogeneous.
If $f(t) \neq 0$, then it is nonhomogeneous.

- To determine the general solution:
  - $\rightarrow$ Find y_h, the homogeneous solution
  - $\rightarrow$ Find y_p, the particular solution

  Then, $y = y_h + y_p$ is the general solution.

- To determine the homogeneous solution $y_h(t)$:

$$r^2 + pr + q = 0 \longleftarrow \text{characteristic equation}$$

  - If r is distinct real roots r_1 & r_2, then

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

  - If r is a repeated real root $r = r_1 = r_2$, then

$$y = C_1 e^{rt} + C_2 t e^{rt}$$

  - If r is a complex conjugate root $r = \alpha + \beta i$

$$y = e^{\alpha t}\left(C_1 \cos(\beta t) + C_2 \sin(\beta t)\right)$$

The Method of Undetermined Coefficients

* This method is a way to solve for $y_p(t)$ by using an Ansatz (or "educated guess") based on the form of $f(t)$.

* These are Ansatz of common functions.

→ If $f(t) = c$, a constant, then $y_p(t) = A_0$, also a constant.

→ If $f(t) = A_n(t)$ where $P_n(t) = P_n t^n + P_{n-1} t^{n-1} + \cdots + P_1 t + P_0$, an nth-order polynomial, then $y_p(t) = A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0 = A_n(t)$, also an nth-order polynomial.
Note that $A_n(t)$ needs to be a complete nth-order polynomial with all different coefficients $A_n, A_{n-1}, \ldots A_1, A_0$, regardles whether $f(t)$ is a complete polynomial.

→ If $f(t) = e^t$, an exponential function, then $y_p(t) = A_0 e^t$, also an exponential.
Note that the exponent in $f(t)$ needs to be the same as the exponent in $y_p(t)$.

→ If $f(t) = \cos(t)$ or $f(t) = \sin(t)$ or $f(t) = \cos(t) + \sin(t)$, then $y_p(t) = A_0 \cos(t) + B_0 \sin(t)$.
Note that the inside function of sin or cos in $f(t)$ need to be the same as the inside function in $y_p(t)$.

→ See <u>table</u> in the worksheet for more common Ansatz functions.

- If when the Ansatz have common terms in $y_h(t)$, then the initial guess of $y_p(t)$ need to be multiplied by t. Check repeatedly until terms of $y_p(t)$ does not have common terms in $y_h(t)$.

* Determining the Coefficients of the Ansatz.

→ If $y_p(t)$ is an Ansatz to $y'' + py' + qy = f(t)$ with undetermined coefficients, the process of determining the coefficients is to take y_p' and y_p'' and plug-in into the ODE, then solve for the coefficients using algebra.

Examples:

① $y'' + y' = 1$

→ Homo Case: $r^2 + r = 0$
$r(r+1) = 0 \rightarrow r_1 = 0$
$r_2 = -1$

$$y_h(t) = C_1 e^{0t} + C_2 e^{-t} = C_1 + C_2 e^{-t}$$ → homogeneous solution

→ Non-homo Case: $f(t) = 1$

"guess" : $y_p = A_1 t$

we multiply by t
because the homo case has a constant

$y_p' = A_1$

$y_p'' = 0$

$y'' + y' = 1 \rightarrow 0 + A_1 = 1 \rightarrow A_1 = 1$

$y_p(t) = t \rightarrow$ particular solution

$\rightarrow y(t) = C_1 + C_2 e^{-t} + t \rightarrow$ general solution

② $y'' - y = \sin(t)$

$\rightarrow$ Homo case: $r^2 - 1 = 0$
$r^2 = 1 \rightarrow r_1 = +1$
$r_2 = -1$

$y_h(t) = C_1 e^{t} + C_2 e^{-t} \rightarrow$ homogeneous solution

$\rightarrow$ Non homo case: $f(t) = \sin(t)$

"guess": $y_p = A\sin(t) + B\cos(t)$

$y_p' = A\cos(t) - B\sin(t)$

$y_p'' = -A\sin(t) - B\cos(t)$

$y'' - y = \sin(t) \rightarrow -A\sin(t) - B\cos(t) - \big(A\sin(t) + B\cos(t)\big) = \sin(t)$
$-2A\sin(t) - 2B\cos(t) = \sin(t)$

$\sin(t)$; $-2A = 1 \rightarrow A = -1/2$
$\cos(t)$; $-2B = 0 \rightarrow B = 0$

So, $y_p(t) = (1/2)\sin(t) \rightarrow$ particular solution

$\rightarrow y(t) = C_1 e^{t} + C_2 e^{-t} + (1/2)\sin(t) \rightarrow$ general solution

③ $y'' - y' - 2y = 3t^2 - 1$

③ $y'' - y' - 2y = 3t^2 - 1$

$\rightarrow$ Homo case: $r^2 - r - 2 = 0$

$$(r-2)(r+1) = 0 \longrightarrow \begin{array}{l} r_1 = 2 \\ r_2 = -1 \end{array}$$

$$y_h(t) = C_1 e^{2t} + C_2 e^{-t}$$

$\rightarrow$ Nonhomo case: $f(t) = 3t^2 - 1$ $\longrightarrow$ 2nd-order polynomial

"guess": $y_p = At^2 + Bt + C$

$y_p' = 2At + B$

$y_p'' = 2A$

$$y'' - y' - 2y = 3t^2 - 1 \longrightarrow \begin{array}{l} 2A - (2At + B) - 2(At^2 + Bt + C) = 3t^2 - 1 \\ 2A - 2At - B - 2At^2 - 2Bt - 2C = 3t^2 - 1 \\ -2At^2 - (2A + 2B)t - 2A - B - 2C = 3t^2 - 1 \end{array}$$

$\downarrow$

$$\left.\begin{array}{rl} t^2: & -2A = 3 \\ t: & -(2A + 2B) = 0 \\ \text{constants}: & -2A - B - 2C = 1 \end{array}\right\} \begin{array}{l} 3 \text{ equations} \\ 3 \text{ unknowns} \end{array}$$

$\downarrow$ convert to matrix vector form

$$\begin{array}{ccc} A & B & C \end{array}$$

$$\underset{3\times 3}{\begin{bmatrix} -2 & 0 & 0 \\ -2 & -2 & 0 \\ -2 & -1 & -2 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} A \\ B \\ C \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}$$

$\downarrow$

Use technology to solve for $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$.

$\downarrow$ wolfram alpha

$$A = -\frac{3}{2}, \quad B = \frac{3}{2}, \quad C = \frac{1}{4}$$

$$y_p(t) = -\frac{3}{2}t^2 + \frac{3}{2}t - \frac{7}{4}$$

$$\longrightarrow y(t) = C_1 e^{2t} + C_2 e^{-t} - \frac{3}{2}t^2 + \frac{3}{2}t - \frac{7}{4}$$

④ $y'' - y' - 2y = 2e^{-3t}$

$\longrightarrow$ Homo case: $r^2 - r - 2 = 0 \longrightarrow \begin{matrix} r_1 = 2 \\ r_2 = -1 \end{matrix} \longrightarrow y_h(t) = C_1 e^{2t} + C_2 e^{-t}$

$\longrightarrow$ Non homo case: $f(t) = 2e^{-3t}$

"guess": $y_p = Ae^{-3t}$

$y'_p = -3Ae^{-3t}$

$y''_p = 9Ae^{-3t}$

$$y'' - y' - 2y = 2e^{-3t} \longrightarrow 9Ae^{-3t} + 3Ae^{-3t} - 2Ae^{-3t} = 2e^{-3t}$$

$$10Ae^{-3t} = 2e^{-3t}$$

$$10A = 2$$

$$A = 1/5$$

$$y_p(t) = \frac{1}{5}e^{-3t}$$

$$\longrightarrow y(t) = C_1 e^{2t} + C_2 e^{-t} + \frac{1}{5}e^{-3t}$$

⑤ $y'' - y' - 2y = 2\cos(3t)$

$\longrightarrow$ Homo case: $y_h(t) = C_1 e^{2t} + C_2 e^{t}$

$\longrightarrow$ non homo case: $f(t) = 2\cos(3t)$

"guess": $y_p = A\cos(3t) + B\sin(3t)$

$y'_p = -3A\sin(3t) + 3B\cos(3t)$

$y''_p = -9A\cos(3t) - 9B\sin(3t)$

$y'' - y' - 2y = 2\cos(3t) \longrightarrow$ simplifications...

$\downarrow$

$$(-11A - 3B)\cos(3t) + (3A - 11B)\sin(3t) = 2\cos(3t)$$

$\downarrow$

$$(-11A-3B)\cos(3t)+(3A-11B)\sin(3t) = 2\cos(3t)$$

$$\downarrow$$

$$\left.\begin{array}{ll}\cos(3t): & -11A-3B = 2\\ \sin(3t): & 3A-11B = 0\end{array}\right\}$$ 2 equations, 2 unknowns

$$\downarrow$$

$$\begin{array}{cc} A & B \end{array}$$
$$\begin{bmatrix}-11 & -3\\ 3 & -11\end{bmatrix}\begin{bmatrix}A\\ B\end{bmatrix} = \begin{bmatrix}2\\ 0\end{bmatrix}$$

use technology to solve for $\begin{bmatrix}A\\ B\end{bmatrix}$

$\downarrow$ wolfy

$$A = -\frac{11}{65},\quad B = -\frac{3}{65}$$

$$y_p(t) = -\frac{11}{65}\cos(3t) - \frac{3}{65}\sin(3t)$$

$$\longrightarrow\ y(t) = C_1 e^{2t} + C_2 e^{-t} - \frac{11}{65}\cos(3t) - \frac{3}{65}\sin(3t)$$

⑥ $y'' - y' - 2y = t^2 e^t$

$\longrightarrow$ Homo Case: $y_h(t) = C_1 e^{2t} + C_2 e^{-t}$

$\longrightarrow$ Non homo case: $f(t) = t^2 e^t \longrightarrow$ polynomial x exponential

"guess": $y_p = (At^2 + Bt + C)e^t$

$$y_p' = e^t\left(At^2 + (2A+B)t + (B+C)\right]$$

$$y_p'' = e^t\left[At^2 + (4A+B)t + (2A+2B+C)\right]$$

$y'' - y' - 2y = t^2 e^t \longrightarrow$ simplifications...

$$e^t\left[t^2(-2A) + t(2A-2B) + (2A+B-2C)\right] = t^2 e^t$$

$$\begin{array}{rl} t^2: & -2A = 1\\ t: & 2A - 2B = 0\\ \text{constant}: & 2A + B - 2C = 0\end{array}$$

$$\downarrow$$

$$\begin{bmatrix}-2 & 0 & 0 \end{bmatrix}\begin{bmatrix}A\end{bmatrix}\begin{bmatrix}1\end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} -2 & 0 & 0 \\ 2 & -2 & 0 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow \text{wolfy}$$

$$A = -\frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = -\frac{3}{4}$$

$$y_p(t) = e^t\left[-\frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4}\right]$$

$$\longrightarrow y(t) = C_1 e^{2t} + C_2 e^{-t} + e^t\left[-\frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4}\right]$$

⑦ $y'' - 2y' + y = 3e^t$

$\longrightarrow$ Homo case: $r^2 - 2r + 1 = 0$

$$(r-1)(r-1) = 0 \longrightarrow r_1 = r_2 = 1$$

$$y_h(t) = C_1 e^t + C_2 t e^t$$

$\longrightarrow$ Non homo case: $f(t) = 3e^t$

"guess": $y_p = At^2e^t$

$$y_p' = 2Ate^t + At^2e^t$$

$$y_p'' = 2Ae^t + 4Ate^t + At^2e^t$$

$y'' - 2y' + y = 3e^t \longrightarrow$ simplifications...

$$2Ae^t = 3e^t$$

$$2A = 3$$

$$A = \frac{3}{2}$$

$$y_p(t) = \frac{3}{2}t^2e^t$$

$$\longrightarrow y(t) = C_1 e^t + C_2 t e^t + \frac{3}{2}t^2 e^t$$