Variation of Parameters for Nonhomogeneous Linear ODEs

Objectives:

1. Introduce the method of Variation of Parameters

Recall: 2nd-order linear ODE with constant coefficients

$$y'' + py' + qy = f(t)$$

where p & q are constants, and $f(t)$ is a function.
If $f(t) = 0$, then it is homogeneous.
If $f(t) \neq 0$, then it is nonhomogeneous.

- To determine the general solution:
  - $\longrightarrow$ Find $y_h(t)$, the homogeneous solution
  - $\longrightarrow$ Find $y_p(t)$, the particular solution

  Then, $y(t) = y_h(t) + y_p(t)$ is the general solution.

- To determine the homogeneous solution $y_h(t)$:

  $r^2 + pr + q = 0 \longrightarrow$ characteristic equation

  - If r is distinct real roots r_1 & r_2, then

    $$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

  - If r is a repeated real root $r = r_1 = r_2$, then

    $$y = C_1 e^{rt} + C_2 t e^{rt}$$

  - If r is a complex conjugate root $r = \alpha + \beta i$

    $$y = e^{\alpha t}\left(C_1 \cos(\beta t) + C_2 \sin(\beta t)\right)$$

The Method of Variation of Parameters

Consider the 2nd-order ODE of the form,

$$y'' + p(t)y' + q(t)y = f(t)$$

where $p(t)$, $q(t)$, and $f(t)$ are functions of t. Note that $p(t)$ & $q(t)$ can be constants.
Assume that $y_1(t)$ and $y_2(t)$ are a set of independent solutions for the homogeneous case

$$y'' + p(t)y' + q(t)y = 0.$$

Then a particular solution to the nonhomogeneous case is,

$$y_p(t) = -y_1 \int \frac{f(t)}{W} y_2 \, dt + y_2 \int \frac{f(t)}{W} y_1 \, dt$$

where the Wronskian W is $W(y_1, y_2) = \det\left(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\right) = y_1 y_2' - y_2 y_1'$.

How is the formula made?

$$\left| \begin{matrix} y_1' & y_2' \end{matrix} \right|$$

<u>How is the formula made?</u>

Suppose that $y_h(t) = y_1(t) + y_2(t)$ is the homogeneous solution to the ODE

$$r(t)y'' + p(t)y' + q(t)y = f(t).$$

We know that $y_1(t)$ & $y_2(t)$ are the independent set of solutions, (also known as the "straight line" solutions).

First, we assume that the form of the particular solution $y_p(t)$ is

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where $u_1(t)$ and $u_2(t)$ are a pair of functions that we need to find.

Next, we do the verification process.

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2$$

$$y_p'' = u_1 y_1'' + u_1' y_1' + u_1' y_1' + u_1'' y_1 + u_2 y_2'' + u_2' y_2' + u_2' y_2' + u_2'' y_2$$

Now, assume $u_1' y_1 = -u_2' y_2$ for some u_1 & u_2.

So, $y_p = u_1 y_1 + u_2 y_2$

$$y_p' = u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'$$

Next, substitute the above back to the ODE.

$$r(t)(u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2') + p(t)(u_1 y_1' + u_2 y_2') + q(t)(u_1 y_1 + u_2 y_2) = f(t)$$

$$r(t)y_1 y_1'' + r(t)u_1' y_1' + r(t)u_2 y_2'' + r(t)u_2' y_2' + p(t)u_1 y_1' + p(t)u_2 y_2' + q(t)u_1 y_1 + q(t)u_2 y_2 = f(t)$$

rearrangings...

$$r(t)(u_1' y_1' + u_2' y_2') + u_1\underbrace{(r(t)y_1'' + p(t)y_1' + q(t)y_1)}_{\text{0 since } y_1 \text{ is one of the homogeneous solution.}} + u_2\underbrace{(r(t)y_2'' + p(t)y_2' + q(t)y_2)}_{\text{0 since } y_2 \text{ is one of the homogeneous solution.}} = f(t)$$

$$r(t)(u_1' y_1' + u_2' y_2') = f(t)$$

$$u_1' y_1' + u_2' y_2' = \frac{f(t)}{r(t)}$$

⟶ for simplicity, let $r(t) = 1$. this just means if $r(t)$ is a function, then $p(t)$ & $q(t)$ needs to be rewritten so that $r(t) = 1$.

So, $u_1' y_1' + u_2' y_2' = f(t)$

Since, $u_1' y_1 = -u_2' y_2$, then we end up with a system of equations:

$$\left.\begin{aligned} \boxed{u_1'}y_1 + \boxed{u_2'}y_2 &= 0 \\ \boxed{u_1'}y_1' + \boxed{u_2'}y_2' &= f(t) \end{aligned}\right\} \text{System of equations for solving } u_1' \text{ \& } u_2'.$$

$$\boxed{u_1'}y_1' + \boxed{u_2'}y_2' = f(t) \Big\}$$ solving u_1' & u_2.

↓ rewrite as matrix-vector form

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

↓ Solve the system of equations through row-reduction or inverses.

$$\left[\begin{array}{cc|c} y_1 & y_2 & 0 \\ y_1' & y_2' & f(t) \end{array}\right] \xrightarrow{R_1^* = (1/y_1)R_1} \left[\begin{array}{cc|c} 1 & y_2/y_1 & 0 \\ y_1' & y_2' & f(t) \end{array}\right]$$

$$\xrightarrow{R_2^* = -y_1'R_1 + R_2} \left[\begin{array}{cc|c} 1 & y_2/y_1 & 0 \\ 0 & -(y_2/y_1)y_1' + y_2' & f(t) \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & y_2/y_1 & 0 \\ 0 & (y_1y_2' - y_2y_1')/y_1 & f(t) \end{array}\right]$$

$$\xrightarrow{R_2^* = \left(\frac{1}{(y_2y_1' + y_1y_2')/y_1}\right)R_2} \left[\begin{array}{cc|c} 1 & y_2/y_1 & 0 \\ 0 & 1 & y_1 f(t)/(y_1y_2' - y_2y_1') \end{array}\right]$$

$$\xrightarrow{R_1^* = (-y_2/y_1)R_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & -y_2 f(t)/(y_1y_2' - y_2y_1') \\ 0 & \boxed{1} & y_1 f(t)/(y_1y_2' - y_2y_1') \end{array}\right]$$

$$u_2' = \frac{y_1 f(t)}{y_1y_2' - y_2y_1'}$$

$$u_1' = \frac{-y_2 f(t)}{y_1y_2' - y_2y_1'}$$

So,

$$u_1' = -y_2 \frac{f(t)}{y_1y_2' - y_2y_1'}$$

$$u_2' = y_1 \frac{f(t)}{y_1y_2' - y_2y_1'}$$

Notice that $y_1y_2' - y_2y_1' = \det\left(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\right) = W(y_1, y_2)$

the <u>Wronskian</u>.

↓ Substitute in W.

$$u_1' = \frac{-y_2 f(t)}{W} \quad \& \quad u_2' = \frac{y_1 f(t)}{W}$$

Now, solving for u_1 & u_2.

$$u_1(t) = \int \frac{-y_2 f(t)}{W}\,dt \qquad u_2(t) = \int \frac{y_1 f(t)}{W}\,dt$$

Therefore, $y_p(t) = u_1 y_1 + u_2 y_2$

or

$$y_p(t) = -y_1 \int \frac{f(t)}{W} y_2\,dt + y_2 \int \frac{f(t)}{W} y_1\,dt,$$

where $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

Examples:

① $y'' - 2y' + y = \dfrac{e^t}{1+t^2}$, solve for the general solution

- Homogeneous solution: $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$\hookrightarrow r_1 = r_2 = 1$ (repeated eigenvalues)

$$y_h(t) = C_1 e^t + C_2 t e^t$$

$\downarrow y_1 = e^t$ $\quad \downarrow y_2 = te^t$

- Nonhomogeneous solution: $f(t) = \dfrac{e^t}{1+t^2}$

$\longrightarrow$ Wronskian: $W(y_1, y_2) = \det\left(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\right)$

$= \det\left(\begin{bmatrix} e^t & te^t \\ e^t & e^t(t+1) \end{bmatrix}\right)$

$= e^t e^t (t+1) - te^t e^t$

$= e^{2t}(t+1) - te^{2t}$

$= e^{2t}(\cancel{t} + 1 - \cancel{t})$

$W = e^{2t}$

$\longrightarrow u_1 = -\displaystyle\int \frac{f(t)}{W} y_2\,dt$

$= -\displaystyle\int \frac{e^t}{1+t^2}\left(\frac{1}{e^{2t}}\right) te^t\,dt$

$= -\displaystyle\int \frac{t\cancel{e^{2t}}}{(1+t^2)\cancel{e^{2t}}}\,dt$

$= -\displaystyle\int \frac{t}{1+t^2}\,dt$

* u-substitution

Let $u = 1+t^2$; $du = 2t\,dt$

$\displaystyle\int \frac{t}{1+t^2}\,dt = \frac{1}{2}\int \frac{1}{u}\,dt$

$= \frac{1}{2}\ln(u)$

$= \frac{1}{2}\ln(1+t^2)$

$u_1 = -\frac{1}{2}\ln(1+t^2)$

$\longrightarrow u_2 = \displaystyle\int \frac{f(t)}{W} y_1\,dt$

$= \displaystyle\int \frac{e^t}{1+t^2}\left(\frac{1}{e^{2t}}\right) e^t\,dt$

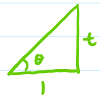
$= \displaystyle\int \frac{\cancel{e^{2t}}}{(1+t^2)\cancel{e^{2t}}}\,dt$

$= \displaystyle\int \frac{1}{\ldots}\,dt$

$$= \int \frac{\cancel{e^{2t}}}{(1+t^2)\cancel{e^{2t}}}\,dt$$

$$= \int \frac{1}{1+t^2}\,dt$$

* Trigonometric substitution



$\tan(\theta) = \frac{t}{1} \longrightarrow \theta = \arctan(t)$

$\sec^2(\theta)\,d\theta = dt$

$$\int \frac{1}{1+t^2}\,dt = \int \frac{1}{1+\tan^2(\theta)}\sec^2(\theta)\,d\theta$$

$$= \int \frac{\sec^2(\theta)}{1+\tan^2(\theta)}\,d\theta$$

$$= \int \frac{\sec^2(\theta)}{\sec^2(\theta)}\,d\theta \longrightarrow \text{since } 1+\tan^2(\theta) = \sec^2(\theta)$$

$$= \int d\theta$$

$$= \theta$$

$$= \arctan(t)$$

$$u_2 = \arctan(t)$$

$\longrightarrow$ Particular solution:

$$y_p(t) = u_1 y_1 + u_2 y_2$$

$$y_p(t) = -\frac{1}{2}\ln(1+t^2)e^t + \arctan(t)\,te^t$$

$$= e^t\left(\arctan(t)\,t - \frac{1}{2}\ln(1+t^2)\right)$$

- General Solution: $y(t) = y_h(t) + y_p(t)$

$$= C_1 e^t + C_2 t e^t + e^t\left(\arctan(t)\,t - \frac{1}{2}\ln(1+t^2)\right)$$

② $t^2y'' - 2ty' + 2y = t\ln(t)$

Suppose we are given that the homogeneous solution is

$$y_h(t) = C_1 t + C_2 t^2$$

Solve for the particular solution $y_p(t)$ and the general solution $y(t)$.

$\longrightarrow$ Since $y_h(t) = C_1 t + C_2 t^2$

$y_1 = t \qquad y_2 = t^2$

$\longrightarrow t^2y'' - 2ty' + 2y = t\ln(t)$

rewrite...

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = \frac{\ln(t)}{t} \longrightarrow f(t) = \frac{\ln(t)}{t}$$

$$\longrightarrow W = \det\left(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\right) = \det\left(\begin{bmatrix} t & t^2 \\ 1 & 2t \end{bmatrix}\right)$$

$$= 2t^2 - t^2$$

$$W = t^2$$

$$\longrightarrow u_1 = -\int \frac{f(t)}{W} y_2 \, dt = -\int \frac{\ln(t)}{t}\left(\frac{1}{t^2}\right) t^2 \, dt$$

$$= -\int \frac{\ln(t)}{t} \, dt$$

$*$ Integration by Parts

Let $u = \ln(t)$ & $dv = 1/t \, dt$

$du = 1/t \, dt \qquad v = \ln(t)$

$$\int \frac{\ln(t)}{t} dt = \ln(t)\ln(t) - \int \frac{\ln(t)}{t} dt$$

$$\int \frac{\ln(t)}{t} dt = (\ln(t))^2 - \int \frac{\ln(t)}{t} dt$$

$$\int \frac{\ln(t)}{t} dt + \int \frac{\ln(t)}{t} dt = (\ln(t))^2$$

$$2\int \frac{\ln(t)}{t} dt = (\ln(t))^2$$

$$\int \frac{\ln(t)}{t} dt = \frac{(\ln(t))^2}{2}$$

$$u_1 = -\frac{(\ln(t))^2}{2}$$

$$\longrightarrow u_2 = \int \frac{f(t)}{W} y_1 \, dt = \int \frac{\ln(t)}{t}\left(\frac{1}{t^2}\right) t \, dt$$

$$= \int \frac{\ln(t)}{t^2} \, dt$$

$*$ Integration by parts

Let $u = \ln(t)$ & $dv = 1/t^2 \, dt$

$du = (1/t) dt \qquad v = -1/t$

$$\int \frac{\ln(t)}{t^2} dt = \ln(t)\left(-\frac{1}{t}\right) - \int \left(-\frac{1}{t}\right)\frac{1}{t} dt$$

$$= -\frac{\ln(t)}{t} + \int \frac{1}{t^2} dt$$

$$= -\frac{\ln(t)}{t} - \frac{1}{t}$$

$$= -\frac{1}{t}(\ln(t) + 1)$$

$$u_2 = -\frac{1}{t}(\ln(t) + 1)$$

$\longrightarrow$ Particular Solution

$$y_p(t) = u_1 y_1 + u_2 y_2$$

$$= t\frac{(\ln(t))^2}{2} - \frac{1}{t}(\ln(t) + 1)t^2$$

$$y_p(t) = -t\frac{(\ln(t))^2}{2} - t(\ln(t) + 1)$$

$\longrightarrow$ General solution

$$y(t) = y_h(t) + y_p(t)$$

$$y(t) = y_h(t) + y_p(t)$$

$$y(t) = C_1 t + C_2 t^2 - \frac{t(\ln(t))^2}{2} - t(\ln(t)+1)$$