Characteristics

Objectives:

1. Show that the inverse of an isomorphism is also an isomorphism.
2. Show that an isomorphism is an equivalence relation (reflexive, symmetric, transitive).
3. Show that vector spaces are isomorphic if and only if they have the same dimension.

Example: Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x,y) = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$.

Isomorphic check for f.

- $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left.\begin{aligned} 2x &= 0 \\ 3y &= 0 \end{aligned}\right\} \text{only solution is } x=0,\ y=0$$

Since $x=0, y=0$ is the only solution to $f(x,y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then f is one-to-one (injective).

- $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 3y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

$$\left.\begin{aligned} 2x &= a \\ 3y &= b \end{aligned}\right\} \text{has at least one solution for any } a \text{ \& } b.$$

Since there is at least one solution for any output value a & b, then f is onto (surjective).

- $$\begin{aligned} f\left(c_1\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) &= f\left(\begin{bmatrix} c_1x_1 \\ c_1y_1 \end{bmatrix} + \begin{bmatrix} c_2x_2 \\ c_2y_2 \end{bmatrix}\right) \\ &= f\left(\begin{bmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \end{bmatrix}\right) \begin{matrix} \leftarrow \text{“}x\text{”} \\ \leftarrow \text{“}y\text{”} \end{matrix} \\ &= \begin{bmatrix} 2(c_1x_1 + c_2x_2) \\ 3(c_1y_1 + c_2y_2) \end{bmatrix} \\ &= \begin{bmatrix} 2c_1x_1 + 2c_2x_2 \\ 3c_1y_1 + 3c_2y_2 \end{bmatrix} \\ &= c_1\begin{bmatrix} 2x_1 \\ 3y_1 \end{bmatrix} + c_2\begin{bmatrix} 2x_2 \\ 3y_2 \end{bmatrix} \\ &= c_1 f\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + c_2 f\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) \end{aligned}$$

This shows closure under addition and scalar multiplication.

- Since f is a bijection and linear, then it is an isomorphism.

The inverse map: $f(x,y) = \begin{bmatrix} 2x \\ 3y \end{bmatrix} \begin{matrix} \longrightarrow & 2x = u \longrightarrow x = u/2 \\ \longrightarrow & 3y = v \longrightarrow y = v/3 \end{matrix}$

$$f^{-1}(u,v) = \begin{bmatrix} u/2 \\ v/3 \end{bmatrix}.$$

Isomorphic check for f^{-1}:

- $f^{-1}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} u/2 \\ v/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\downarrow$$

$$\left.\begin{matrix} u/2 = 0 \\ v/2 = 0 \end{matrix}\right\} \text{the only solution is } u=0,\ v=0.$$

Since $u=0, v=0$ is the only solution to $f^{-1}(u,v) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then f^{-1} is one-to-one (injective).

- $f^{-1}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} u/2 \\ v/3 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

$$\downarrow$$

$$\left.\begin{matrix} u/2 = a \\ v/3 = b \end{matrix}\right\} \text{There is at least one solution for any } a \text{ \& } b.$$

Since there is at least one solution for any a & b for $f(x,y) = \begin{pmatrix} a \\ b \end{pmatrix}$, then f^{-1} is onto (surjective).

- $f^{-1}\left(c_1 \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} + c_2 \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} c_1 u_1 \\ c_1 v_1 \end{bmatrix} + \begin{bmatrix} c_2 u_2 \\ c_2 v_2 \end{bmatrix}\right)$

$$= f\left(\begin{bmatrix} c_1 u_1 + c_2 u_2 \\ c_1 v_1 + c_2 v_2 \end{bmatrix}\right) \begin{matrix} \longleftarrow \text{"u"} \\ \longleftarrow \text{"v"} \end{matrix}$$

$$= \begin{bmatrix} (c_1 u_1 + c_2 u_2)/2 \\ (c_2 v_1 + c_2 v_2)/3 \end{bmatrix}$$

$$= c_1 \begin{bmatrix} u_1/2 \\ v_1/3 \end{bmatrix} + c_2 \begin{bmatrix} u_2/2 \\ v_2/3 \end{bmatrix}$$

$$= c_1 f\left(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}\right) + c_2 f\left(\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\right)$$

this shows closure under addition and scalar multiplication.

- Since f^{-1} is bijective and linear, then it is isomorphic.

* <u>If f is an isomorphism, then the inverse map f^{-1} is also an isomorphism.</u>

Let $f: V \to W$ be an isomorphic mapping from vector spaces V to W.
Suppose that $\vec{w}_1, \vec{w}_2 \in W$. Since f is an isomorphism, then $\vec{w_1} = f(\vec{v_1})$ & $\vec{w_2} = f(\vec{v_2})$ for $\vec{v}_1, \vec{v}_2 \in V$.
So, $f^{-1}(c_1 \vec{w_1} + c_2 \vec{w_2}) = f^{-1}(c_1 f(\vec{v_1}) + c_2 f(\vec{v_2}))$

$$= f^{-1}(f(c_1 \vec{v_1} + c_2 \vec{v_2}))$$

$$= c_1 \vec{v_1} + c_2 \vec{v_2}$$

$$= c_1 f^{-1}(\vec{w_1}) + c_2 f^{-1}(\vec{w_2})$$

$$= f^{-1}(f(c_1\vec{v_1}+c_2\vec{v_2}))$$
$$= c_1\vec{v_1}+c_2\vec{v_2}$$
$$= c_1 f^{-1}(\vec{w_1})+c_2 f^{-1}(\vec{w_2})$$

Since $f^{-1}(\vec{w_1})=\vec{v_1}$ and $f^{-1}(\vec{w_2})=\vec{v_2}$.

Thus, f^{-1} is closed under addition and scalar multiplication.

Since f is bijective, then the correspondence between the sets should be one-to-one and onto, which means that f^{-1} is also bijective.

Therefore, f^{-1} is isomorphic.

* Isomorphism is an equivalence relation between vector spaces.

- symmetry: If V is isomorphic to W, then W is also isomorphic to V since $f: V \to W$ & $f^{-1}: V - W$ are both isomorphic.

- reflexive: $id(c_1\vec{v_1}+c_2\vec{v_2}) = c_1\vec{v_1}+c_2\vec{v_2}$

$$= c_1 id(\vec{v_1}) + c_2 id(\vec{v_2})$$

where $id(\vec{v_1}) = \vec{v_1}$ is the identity map.

- transitive: Let $f: V \to W$ and $g: W \to V$ be isomorphic maps.

Suppose $g \circ f: V \to U$ be the composition of f and g.

A bijective f and g yeild a bijective $g \circ f$.

Now, we check for linearity (closure under addition and scalar multiplication).

$$g\circ f(c_1\vec{v_1}+c_2\vec{v_2}) = g(f(c_1\vec{v_1}+c_2\vec{v_2}))$$
$$= g(c_1 f(\vec{v_1}) + c_2 f(\vec{v_2}))$$
$$= c_1 g(f(\vec{v_1})) + c_2 g(f(\vec{v_2}))$$
$$= c_1 (g\circ f)(\vec{v_1}) + c_2 (g\circ f)(\vec{v_2}).$$

thus $g \circ f$ is an isomorphism.

* Vector spaces are isomorphic if and only if they have the same dimension.

$\longrightarrow$ If spaces are isomorphic then they have the same dimension.

Let $f: V \to W$ be an isomorphism with basis $B = \{\vec{\beta_1}, \dots, \vec{\beta_n}\}$ for V and $D = \{f(\vec{\beta_1}), \dots, f(\vec{\beta_n})\}$ for W.

Let $\vec{w} \in W$. Since f is an isomorphism it is onto and so there is a $\vec{v} \in V$ with $\vec{w} = f(\vec{v})$.

So,

$$\vec{w} = f(\vec{v}) = f(v_1\vec{\beta_1} + \cdots + v_n\vec{\beta_n})$$
$$= v_1 f(\vec{\beta_1}) + \cdots + v_n f(\vec{\beta_n}).$$

If $\vec{0}_W = c_1 f(\vec{\beta_1}) + \cdots + c_n f(\vec{\beta_n})$

$$= f(c_1\vec{\beta_1} + \cdots + c_n\vec{\beta_n}),$$

then, since f is one-to-one and so the only

$= f(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n)$,

then, since f is one-to-one and so the only vector sent to $\vec{0}_W$ is $\vec{0}_V$, we have

$\vec{0}_V = c_1\vec{\beta}_1 + \cdots + c_1\vec{\beta}_n$.

So, all $c_1, \ldots, c_n$ has to equal zero. ⟵ this shows they have the same number of basis vectors.

⟶ If spaces have the same dimension then they are isomorphic.

Let V be an n-dimensional vector space with basis vectors $B = \{\vec{\beta}_1, \ldots, \vec{\beta}_n\}$.

Consider

$\vec{v} = v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n$ ⟵ $\mathrm{Rep}_B(\vec{v})$

If $\mathrm{Rep}_B(u_1\vec{\beta}_1 + \cdots + u_n\vec{\beta}_n) = \mathrm{Rep}_B(v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n)$, then $\vec{u} = \vec{v}$ and so $u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$. } one-to-one

Any member of $\mathbb{R}^n$: $\vec{w}$ is the image of some $\vec{v} \in V$, that is $\vec{w} = \mathrm{Rep}_B(w_1\vec{\beta}_1 + \cdots + w_n\vec{\beta}_n)$. } onto

Structure preservation:

$$\mathrm{Rep}_B(r\vec{u} + s\vec{v}) = \mathrm{Rep}_B\big((ru_1+sv_1)\vec{\beta}_1 + \cdots + (ru_n+sv_n)\vec{\beta}_n\big)$$

$$= \begin{bmatrix} ru_1 + sv_1 \\ \vdots \\ ru_n + sv_n \end{bmatrix}$$

$$= r\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + s\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= r\,\mathrm{Rep}_B(\vec{u}) + s\,\mathrm{Rep}_B(\vec{v})$$ ⟵ shows closure under addition and scalar multiplication.

Examples:

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{f} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is isomorphic

because $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\vec{\beta}_1 + b\vec{\beta}_2 + c\vec{\beta}_3 + d\vec{\beta}_4 \xrightarrow{f} a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3 + d\vec{e}_4 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

four elements

four elements

or $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{f} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ shows a bijection and it is linear.

- $f: \mathbb{R} \to \mathbb{R}^2$ is not isomorphic since they have different dimensions.

- $f: P_5 \to \mathbb{R}^5$ is not isomorphic since

$$P_5 = \{a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \mid \underbrace{a_0, a_1, a_2, a_3, a_4, a_5}_{6 \text{ elements}} \in \mathbb{R}\}$$

but $\mathbb{R}^5$ has vectors with 5 elements. So, they don't have the same dimensions.

- $f: P_3 \to \mathbb{R}^4$ is isomorphic since

$$P_3 = \{a_3x^3 + a_2x^2 + a_1x^1 + a_0 \mid \underbrace{a_0, a_1, a_2, a_3}_{4 \text{ elements}} \in \mathbb{R}\}$$

and $\mathbb{R}^4$ has vectors with 4 elements. So, they have the same dimensions.