Dimension

Objectives

1. Define homogeneous and non-homogeneous systems.
2. Define "dimensions" of a vector space.
3. Define the "dimensions" of a solution set.

Recall: Matrix-Vector Form of Linear Equations

A linear system of m equations and n variable,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

can be written in matrix-vector form:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b}}$$

or $A\vec{x} = \vec{b}$.

- A is size $m \times n$
- $\vec{x}$ is size $n \times 1$
- $\vec{b}$ is size $m \times 1$

Homogeneous vs Non-homogeneous Equations

Given a linear equation $A\vec{x} = \vec{b}$,

- if $\vec{b} = \vec{0}$, then $A\vec{x} = \vec{0}$ is a homogeneous system.
- if $\vec{b} \neq \vec{0}$, then $A\vec{x} = \vec{b}$ is a non-homogeneous system.

* Example: Given the linear system in $\mathbb{R}^4$,

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}}_{\substack{A \\ 4\times 4}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\substack{\vec{x} \\ 4\times 1}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\substack{\vec{b} \\ 4\times 1}}$$

Here, $\vec{b} = \vec{0}$, so, a homogeneous system.

Solution set:

$$\xrightarrow{\text{RREF}} \begin{array}{c} \begin{array}{cccc|c} & & x_3 & x_4 & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & \boxed{1} & \boxed{1} & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

x_3 & x_4 are free variables

Let $x_3 = s$ and $x_4 = t$.

the $\vec{0}$ vector (always for homo equations)

$$\left.\begin{aligned} x_1 + s + t &= 0 \\ x_2 + s - t &= 0 \\ x_3 &= s \\ x_4 &= t \end{aligned}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \underbrace{s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{\text{parametric equation for a plane}}$$

So, the basis of the solution set is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, which is also in the subspace of $\mathbb{R}^4$ since it passes through the origin.

The <u>dimension</u> of this solution set is 2, since there are two vectors in the solution set basis.

<u>Vector Space Dimensions</u>

Vector Space Dimensions

The dimension of a vector space is the number of vectors in its basis.

* Examples:
1. $\mathbb{R}^1$ has one dimension, only one vector in its basis
2. $\mathbb{R}^2$ has two dimension, only two vectors in its basis
3. $\mathbb{R}^3$ has three dimension, only three vectors in its basis

Solution Set Dimensions

The dimensions of a solution set is the number of vectors in its basis, which is always less than the dimension of the linear system.

* Example: Given a linear system $A\vec{x} = \vec{b}$ in $\mathbb{R}^3$ and its solution set with 2 basis vectors, then the dimension of the solution set is 2.