Echelon Form

Objectives:

1. Layout Echelon forms
2. Underdetermined and overdetermined systems

General form of Linear Systems

Consider the system of m linear equations with n variables $x_1, x_2, \dots, x_n$ in general form:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

where a_{ij} and b_i are constants for $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$.

General form of Augmented Matrices

$$\begin{array}{c}
\begin{array}{cccc} x_1 & x_2 & & x_n \end{array} \\
\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}\right]
\end{array}$$

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Row Echelon Forms (Examples & non-exhaustive)

* Row echelon forms are the end goal of Gaussian elimination of augmented matrices.

* Case 1: $m = n$; # of equation = # of variables

$\longrightarrow m = 2$: $\left[\begin{array}{cc|c} 1 & * & * \\ 0 & 1 & * \end{array}\right] \longrightarrow$ unique solution

pivots

$$\left[\begin{array}{cc|c} 1 & * & * \\ 0 & 0 & * \end{array}\right] \qquad \left[\begin{array}{cc|c} 0 & 0 & * \\ 0 & 0 & 0 \end{array}\right] \Bigg\} \text{ no solution}$$

$$\begin{array}{c} \begin{array}{ccc} & x_2 & \end{array} \\ \left[\begin{array}{cc|c} 1 & * & * \\ 0 & 0 & 0 \end{array}\right] \end{array} \Big\} \text{ infinite solutions}$$

$$\begin{array}{c} \begin{array}{ccc} & x_2 & \end{array} \\ \left[\begin{array}{cc|c} 1 & * & * \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{ccc} x_1 & & \end{array} \\ \left[\begin{array}{cc|c} 0 & 1 & * \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

} infinite solutions
* columns without pivots are free variables

$\longrightarrow$ $m=3$:

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right] \longrightarrow \text{unique solution}$$

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & * \end{array}\right] \longrightarrow \text{no solution}$$

$$\begin{array}{c} \begin{array}{cccc} & x_2 & & \end{array} \\ \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} x_1 & & & \end{array} \\ \left[\begin{array}{ccc|c} 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & & \end{array} \\ \left[\begin{array}{ccc|c} 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} x_1 & & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

} Infinite solutions
* columns w/o pivots are free variables

* Case 2: $m > n$; more equations than variables (overdetermined systems)

$\longrightarrow$ $m=3$;

$$\left[\begin{array}{cc|c} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \text{unique solution}$$

$$\left[\begin{array}{cc|c} 1 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \text{no solution (this happens often)}$$

$$\begin{array}{c} \begin{array}{ccc} & x_2 & \end{array} \\ \left[\begin{array}{cc|c} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{ccc} x_1 & & \end{array} \\ \left[\begin{array}{cc|c} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

infinite solutions
* columns w/o pivots are free variables

* Case 3: $m < n$; more variables than equations (<u>underdetermined systems</u>)

$\longrightarrow m = 2, n = 3;$

$$\begin{array}{c} \begin{array}{cccc} & & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} & x_2 & & \end{array} \\ \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 1 & * \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} x_1 & & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} x_1 & & & \end{array} \\ \left[\begin{array}{ccc|c} 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

infinite solutions (this happens often)
* columns w/o pivots are free variables

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 0 & * \end{array}\right] \longrightarrow \text{no solution}$$

No unique solutions for underdetermined systems

* Summary: Row echelon form is a way of arranging a matrix so that each nonzero row starts with a leading entry (1st non-zero number) that is to the right of the leading entry in the row above, and all rows of zeros are placed at the bottom.
A pivot is one of these leading entries that anchors a row and helps determine the structure of the solution set.