Homomorphisms

Objectives:

1. Define homomorphisms
2. Show homomorphic transformations
3. Show that isomorphism is a special case of homomorphism

Recall: Isomorphism

$f: V \to W$ is isomorphic if V and W have the same dimension and

1. it is closed under addition and scalar multiplication
2. it is bijective (one-to-one and onto).

* Homomorphism

A structure preserving map between two vector spaces that does not have to be the same size.

Let V and W be vector spaces and $f: V \to W$.

A function f is called a homomorphism (also known as a linear map) if for all $\vec{v}_1, \vec{v}_2 \in V$ and scalars $c \in \mathbb{R}$:

- $f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$
- $f(c\vec{v}_1) = cf(\vec{v}_1)$.

If $f: V \to W$ is bijective, then it is isomorphic.

Examples:

1. $$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y+z \end{bmatrix}$$

Let $f: \mathbb{R}^3 \to \mathbb{R}^2$.

Let $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in \mathbb{R}^3$ and $c_1, c_2 \in \mathbb{R}$.

$$f\left(c_1\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + c_2\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} c_1x_1 \\ c_1y_1 \\ c_1z_1 \end{bmatrix} + \begin{bmatrix} c_2x_2 \\ c_2y_2 \\ c_2z_2 \end{bmatrix}\right)$$

$$= f\left(\begin{bmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \\ c_1z_1 + c_2z_2 \end{bmatrix}\right) \begin{matrix} \leftarrow \text{“}x\text{”} \\ \leftarrow \text{“}y\text{”} \\ \leftarrow \text{“}z\text{”} \end{matrix}$$

$$= \begin{bmatrix} c_1x_1 + c_2x_2 \\ c_1x_1 + c_2x_2 + c_1y_1 + c_2y_2 + c_1z_1 + c_2z_2 \end{bmatrix}$$

$$= \begin{bmatrix} c_1x_1 + c_2x_2 \\ c_1x_1 + c_1y_1 + c_1z_1 + c_2x_2 + c_2y_2 + c_2z_2 \end{bmatrix}$$

$$= \begin{bmatrix} c_1x_1 \\ c_1x_1 + c_1y_1 + c_1z_1 \end{bmatrix} + \begin{bmatrix} c_2x_2 \\ c_2x_2 + c_2y_2 + c_2z_2 \end{bmatrix}$$

$$= c_1 \begin{bmatrix} x_1 \\ x_1 + y_1 + z_1 \end{bmatrix} + c_2 \begin{bmatrix} x_2 \\ x_2 + y_2 + z_2 \end{bmatrix}$$

$$= c_1 f\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + c_2 f\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right)$$ ← this shows closure under addition and scalar multiplication

So, f is homomorphic.

2. $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let $f: \mathbb{R}^3 \to \mathbb{R}^2$.

Let $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in \mathbb{R}^3$ and $c_1, c_2 \in \mathbb{R}$.

$$f\left(c_1 \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + c_2 \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \\ c_1z_1 + c_2z_2 \end{bmatrix}\right) \begin{matrix} \leftarrow \text{"x"} \\ \leftarrow \text{"y"} \\ \leftarrow \text{"z"} \end{matrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ ← always $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So, if $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then $f\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Thus, f is not homomorphic because $\vec{0} \in \mathbb{R}^3$ must map to $\vec{0} \in \mathbb{R}^2$.

* A linear map sends the zero vector to the zero vector.

Example: $\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{f} \begin{bmatrix} x \\ 0 \end{bmatrix}$ is a homomorphism

because you can show closure under addition and scalar multiplication and $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. ← $\vec{0}$ maps to $\vec{0}$

because you can show closure under addition and scalar multiplication and $\begin{bmatrix}0\\0\end{bmatrix} \rightarrow \begin{bmatrix}0\\0\end{bmatrix}$. $\longleftarrow$ $\vec{0}$ maps to $\vec{0}$

* **A homomorphism is determined by its action on a basis.**

If V is a vector space with basis $\{\vec{\beta_1}, \dots \vec{\beta_n}\}$, if W is a vector space, and if $\vec{w_1}, \dots, \vec{w_n} \in W$, then there exist a homomorphism from V to W sending each $\vec{\beta_i}$ to $\vec{w_i}$ and it is unique.

* **Linear Extension**

Let V and W be vector spaces and let $B = \{\vec{\beta_1}, \dots, \vec{\beta_n}\}$ be a basis for V.

A function defined on that basis $f: B \rightarrow W$ is extendend linearly to a function $\hat{f}: V \rightarrow W$ if for all $\vec{v} \in V$ such that $\vec{v} = c_1\vec{\beta_1} + \dots + c_n\vec{\beta_n}$, the action map is

$$\hat{f}(\vec{v}) = c_1 f(\vec{\beta_1}) + \dots + c_n f(\vec{\beta_n}).$$

Example: Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with basis $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$.

$$f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\1\end{bmatrix}, \quad f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-4\\4\end{bmatrix} \quad \longleftarrow \text{defined action}$$

$$f\left(\begin{bmatrix}3\\-2\end{bmatrix}\right) = f\left(\underbrace{3\begin{bmatrix}1\\0\end{bmatrix} - 2\begin{bmatrix}0\\1\end{bmatrix}}_{\text{linear combination}}\right) = 3f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) - 2f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= 3\begin{bmatrix}-1\\1\end{bmatrix} - 2\begin{bmatrix}-4\\4\end{bmatrix}$$

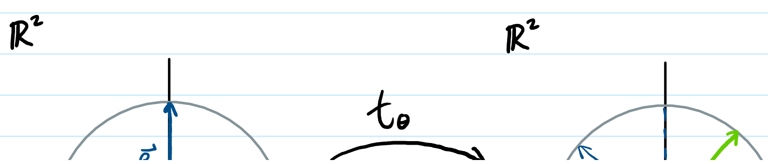
$$= \begin{bmatrix}-3+8\\3-8\end{bmatrix}$$

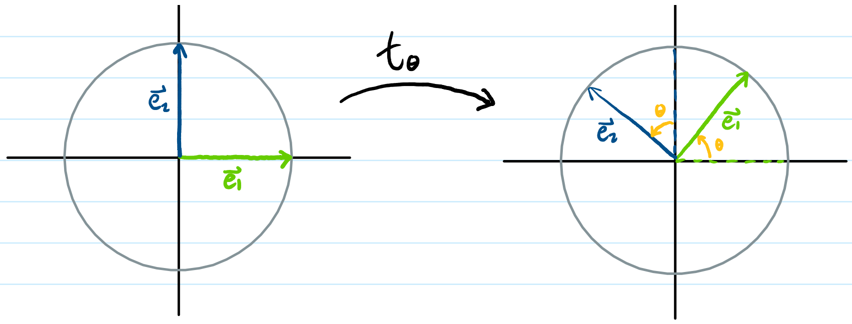
$$= \begin{bmatrix}5\\-5\end{bmatrix}$$

* **A linear map from a space into itself $f: V \rightarrow W$ is a linear transformation.**

Example: Rotation Transformations

Let $f_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear map that rotates vectors in $\mathbb{R}^2$ back in $\mathbb{R}^2$ counterclockwise by θ.




t_θ
$\vec{e}_1$
$\vec{e}_2$
θ