Isomorphisms

Objectives:

1. Define a linear transformation
2. Define an isomorphism
3. Determine if a transformation is isomorphic

Linear Transformation

* Let V and W be two vector spaces.

  A mapping $f: V \to W$ is a called a linear transformation or a linear map if it preserves the algebriac operations of addition and scalar multiplication.

  Specifically, for vectors $\vec{v}_1$ & $\vec{v}_2 \in V$ and $c \in \mathbb{R}$:

  - $f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$ ⟵ closure under addition
  - $f(c\vec{v}_1) = cf(\vec{v}_1)$ ⟵ closure under scalar multiplication

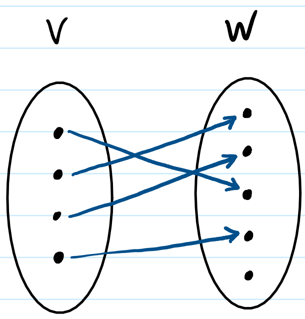
Isomorphism

* An isomorphism between two vector spaces V and W is a map $f: V \to W$ that

  - f is one-to-one and onto (also called a bijection).
  - f preserves structure, meaning if $\vec{v}_1, \vec{v}_2 \in V$ and $c \in \mathbb{R}$, then
    1. $f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$.
    2. $f(c\vec{v}_1) = cf(\vec{v}_2)$.

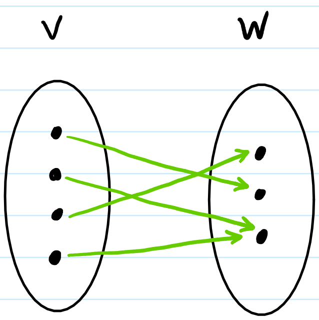
Bijection

* A bijection is both injective (one-to-one) and surjective (onto).



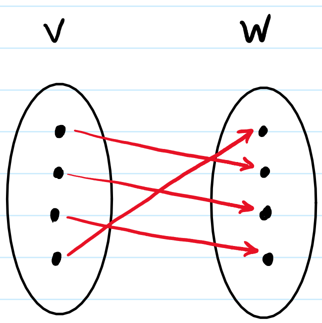
Injective
(one-to-one)

- Each element in V is mapped to a distict element in W.



Surjective
(onto)

- Each element in V is mapped to an element in W.



Bijection
(one-to-one and onto)

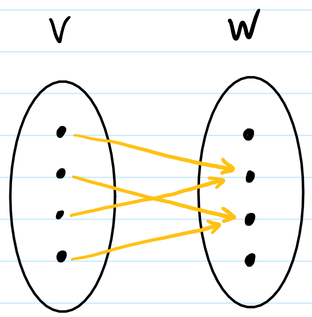
- Each element in V has exactly one element in W.

• Each element in V is mapped to a distict element in W.
• W can have more elements the V, but every element in V must map to at most one element in W.
• Size of W $\geq$ size of V.

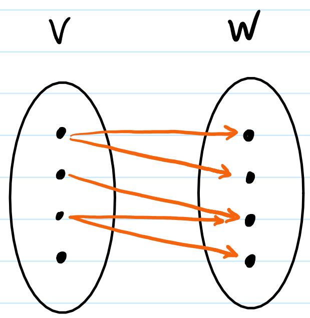
• Each element in V is mapped to an element in W.
• Each element in V have at least one mapped element in W.
• Size of W $\leq$ size of V.

• Each element in V has exactly one element in W.
• Every element in V is mapped to every element in W.
• Size of W = Size of V

Neither one-to-one nor onto



Neither injective nor surjective



Not a function

Example:

Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$$

Show f is an isomorphism.

• Bijection:

$\longrightarrow$ Injection: Show $f = \vec{0}$ for $x=0, y=0$ only.

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\downarrow$

$$\left.\begin{matrix} x+y=0 \\ x-y=0 \end{matrix}\right\}$$ system of equations

$\downarrow$ augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$\downarrow$ RREF

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \longrightarrow \begin{matrix} x=0 \\ y=0 \end{matrix}$$

$$\begin{array}{cc} & \downarrow \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \\ x & y \end{array} \longrightarrow \begin{array}{l} x=0 \\ y=0 \end{array}$$

Since $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ the only solution for $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

then it is injective (one-to-one).

→ Surjection: Show that $f = \begin{bmatrix} a \\ b \end{bmatrix}$ will have at least one solution.

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\downarrow$$

$$\left.\begin{array}{l} x+y = a \\ x-y = b \end{array}\right\} \text{system of equations}$$

$$\Big\downarrow \text{ augmented matrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & -1 & b \end{array}\right]$$

$$\Big\downarrow \text{ RREF}$$

$$\left[\begin{array}{cc|c} 1 & 0 & (a+b)/2 \\ 0 & 1 & (a-b)/2 \end{array}\right] \longrightarrow \text{has a solution for all } a, b \text{ values.}$$

Thus, it is surjective (onto).

Since f is one-to-one and onto, then it is a bijection.

- Linear transformation: Let $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ and $c_1, c_2 \in \mathbb{R}$

$$f\left(c_1\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} c_1x_1 \\ c_1y_1 \end{bmatrix} + \begin{bmatrix} c_2x_2 \\ c_2y_2 \end{bmatrix}\right)$$

$$= f\left(\begin{bmatrix} (c_1x_1 + c_2x_2) \\ (c_1y_1 + c_2y_2) \end{bmatrix}\right) \begin{array}{l} \longleftarrow \text{“}x\text{”} \\ \longleftarrow \text{“}y\text{”} \end{array}$$

$$= \begin{bmatrix} (c_1x_1 + c_2x_2) + (c_1y_1 + c_2y_2) \\ (c_1x_1 + c_2x_2) - (c_1y_1 + c_2y_2) \end{bmatrix}$$

$$= \begin{bmatrix} (c_1x_1 + c_1y_1) + (c_2x_2 + c_2y_2) \\ (c_1x_1 - c_1y_1) + (c_2x_2 - c_2y_2) \end{bmatrix}$$

$$= \begin{bmatrix} c_1(x_1+y_1) \\ c_1(x_1-y_1) \end{bmatrix} + \begin{bmatrix} c_2(x_2+y_2) \\ c_2(x_2-y_2) \end{bmatrix}$$

$$= c_1 \begin{bmatrix} x_1+y_1 \\ x_1-y_1 \end{bmatrix} + c_2 \begin{bmatrix} x_2+y_2 \\ x_2-y_2 \end{bmatrix}$$

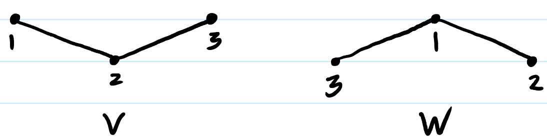
$$= c_1 f\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + c_2 f\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) \longleftarrow$$ This shows closure under addition and scalar multiplication.

Therefore, f is f is a linear mapping.

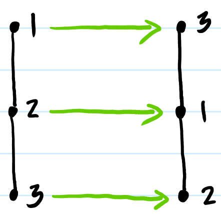
Since f is bijective and a linear map, then f is an isomorphic transformation.

Application to Networks

Example:



→ Networks V and W are isomorphic through a node label permutation mapping.



bijective and preserves structure

f: $1 \to 3$
$2 \to 1$
$3 \to 2$

→ Adjecency matrices

$$V = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \qquad W = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

permutation matrix:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$ using the map f.

Check: If V and W are isomorphic then $W = PVP^T$.

$$P^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$PAP^T = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$W = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \checkmark$$