Linear Combination

Objectives:

1. Introduce vector notation
2. Define a linear combination
3. System of equations as a linear combination

Vector Notation

* A <u>column vector</u> in $\mathbb{R}^n$ is written as a column of numbers:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \longrightarrow$$ sometimes $\vec{v}$ is written in bold $\mathbf{v}$. sometimes ( ) is used, instead of [ ].

Here:

- $v_1, v_2, \dots, v_n$ are called the <u>elements</u> (entries) of the vector.
- the dimension of the vector is the number of entries (e.g., n entries $\longrightarrow$ vector in $\mathbb{R}^n$).
- We assume all vectors are column vectors, unless it says otherwise.

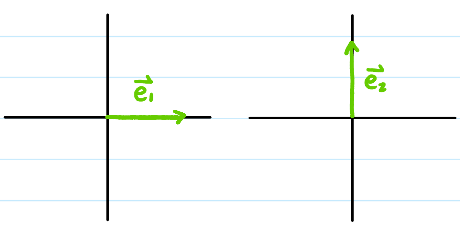
* A <u>row vector</u> is the <u>transpose</u> of the column vector:

$$\vec{v}^T = [v_1 \ v_2 \ \cdots \ v_n].$$

Standard Basis Vectors

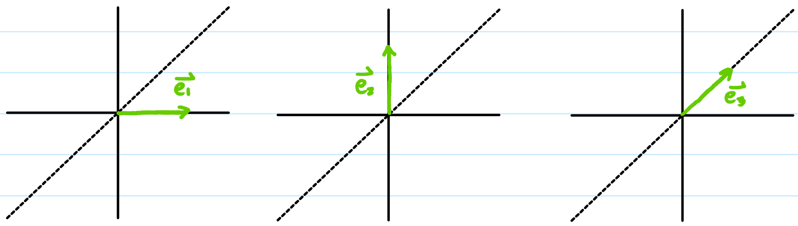
In $\mathbb{R}^n$, we often use <u>standard basis vectors</u> to describe directions:

* In $\mathbb{R}^2$: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



* In $\mathbb{R}^3$:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



* In $\mathbb{R}^n$:

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad \vec{e_n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Linear Combination

Definition (linear combination): Let $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ be vectors in $\mathbb{R}^n$.

A linear combination of these vectors is any vector of the form:

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k}$$

where $c_1, c_2, \ldots, c_k \in \mathbb{R}$ are called scalars (coefficients).

Example 1 (in $\mathbb{R}^2$)

* Let $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Linear combination: $c_1 v_1 + c_2 v_2 = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + 3c_2 \\ 2c_1 + c_2 \end{bmatrix}$.

Definition (standard basis vectors): Any vector in $\mathbb{R}^n$ can be written as a linear combination of the standard basis vectors:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + v_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$= v_1\vec{e_1} + v_2\vec{e_2} + \cdots + v_n\vec{e_n}$$

$$\vec{v} = \sum_{i=1}^{n} v_i\vec{e_i}.$$

Example 2: (in $\mathbb{R}^2$)

* Let $\vec{v} = [v_1]$, rewritten as $v_1\vec{e_1} + v_2\vec{e_2} = v_1[1] + v_2[0] = [v_1]$.

* Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, rewritten as $v_1\vec{e}_1 + v_2\vec{e}_2 = v_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

Example 3: (in $\mathbb{R}^3$)

* Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, rewritten as $v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3 = v_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.

Dot Product of Two Vectors

Definition (Dot product): Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$.

The dot product (also called the inner product) of $\vec{u}$ and $\vec{v}$ is defined as:

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \sum_{i=1}^{n} u_iv_i .$$

System of Equations as a Linear Combination

* Consider a linear system with m equations and n variables.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \quad \xrightarrow{\text{dot product}} \quad \vec{a}_{2n}\cdot\vec{x} = b_2 \text{ where } \vec{a}_{2n} = \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

can be written in matrix-vector form:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b}}$$

or $A\vec{x} = \vec{b}$.

* Linear system as a linear combination of $\vec{b}$

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$\vec{b}$

columns of A

Example 4: $\begin{aligned} x_1 + 2x_2 &= 5 \\ 3x_1 + x_2 &= 4 \end{aligned}$ $\xrightarrow{\text{augmented matrix}}$ $\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & \text{"="} \end{array} \\ \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 1 & 4 \end{array}\right] \end{array}$

$\xrightarrow{\text{matrix-vector form}}$ $\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 5 \\ 4 \end{bmatrix}}$

Solving is asking: Can $\vec{b}$ be written as a linear combination of the columns of A?

What values of $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, so that $x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$?

$\xrightarrow{\text{Gauss-Jordan Elimination}}$ $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 1 & 4 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \end{array}$

$\xrightarrow{R_2 = -3R_1 + R_2}$ $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -11 \end{array}\right]$

$\xrightarrow{R_2 = (-1/5)R_2}$ $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 11/5 \end{array}\right]$ ref

$\xrightarrow{R_1 = -2R_2 + R_1}$ $\begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 3/5 \\ 0 & 1 & 11/5 \end{array}\right] \end{array}$ rref

$\begin{aligned} x_1 &= 3/5 \\ x_2 &= 11/5 \end{aligned}$ unique solution.

So, $\vec{x} = \begin{bmatrix} 3/5 \\ 11/5 \end{bmatrix}$. Thus, $\vec{b} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ can be written a linear combination of the columns of A.

Example 5: Express $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ as a linear combination of $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

$c_1\vec{v}_1 + c_2\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \longrightarrow c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$\xrightarrow{\text{matrix-vector form}}$ $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$$\xrightarrow{\text{augmented matrix}} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 3 & 5 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \end{array}$$

$$\xrightarrow{R_2 = -R_1 + R_2} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \end{array}\right] \quad \text{ref}$$

$$\xrightarrow{R_1 = (-2)R_2 + R_1} \begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \end{array} \quad \text{rref}$$

$$\downarrow$$

$c_1 = 2$ unique solution

$c_2 = 1$

So, $\begin{bmatrix} 4 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.