Linear Independence

Objectives

1. Define linear independence
2. Define a basis and what it means visually

Example 1: Are vectors $\vec{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ linearly independent?

We need $\underbrace{c_1\vec{v_1} + c_2\vec{v_2}}_{\text{Span}\{\vec{v_1},\, \vec{v_2}\}} = \vec{0}$

$$c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{aligned} c_1 - c_2 &= 0 \\ -c_1 + c_2 &= 0 \end{aligned}$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & \boxed{-1} & 0 \\ 0 & \boxed{0} & 0 \end{array}\right] \quad c_1 = c_2$$

$\hookrightarrow c_2$ as free variable

infinite solutions

Since $c_1 = c_2 \neq 0$, then $\vec{v_1}$ and $\vec{v_2}$ are <u>linearly dependent</u>.

Example 2: Are vectors $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ linearly independent?

We need $c_1\vec{v_1} + c_2\vec{v_2} = \vec{0}$

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{aligned} c_1 + 3c_2 &= 0 \\ c_1 \quad\quad &= 0 \end{aligned}$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \end{array}\right]$$

unique solution

$c_1 = c_2 = 0$

Since $c_1 = c_2 = 0$, then $\vec{v_1}$ and $\vec{v_2}$ are <u>linearly independent</u>.

* <u>Linear Independence</u>

For vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$. They are <u>linearly independent</u> if and only if the only scalars $c_1, c_2, \ldots, c_k$ such that
$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_3\vec{v_3} = \vec{0}$ are $c_1 = c_2 = \cdots = c_k = 0$.

In other words, $\sum_{i=1}^{k} c_i\vec{v_i} = \vec{0}$ is only true if $c_i = 0$ for all $i = \{1, 2, \ldots k\}$.

Example 3: Are vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ linearly independent?

We need $\underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3}_{\text{Span}\{v_1,\, v_2,\, v_3\}} = \vec{0}$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{aligned} c_1 - 2c_2 &= 0 \\ c_1 - c_3 &= 0 \\ 2c_2 + c_1 &= 0 \end{aligned}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{c|c} \boxed{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{array}\right]$$

unique solution

$c_1 = c_2 = c_3 = 0$

Since $c_1 = c_2 = c_3 = 0$, then $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are linearly independent.

Example 4: Using the vectors from Example 3, is $\vec{v}_1, \vec{v}_2, \vec{v}_3$ a <u>basis</u> of $\mathbb{R}^3$?

- Let $\mathbb{R}^3$ be a vector space and $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$.
- Span needs to be in $\mathbb{R}^3$: $\text{Span}\{v_1, v_2, v_3\} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{v} \in \mathbb{R}^3$
- $\vec{v}_1, \vec{v}_2, \vec{v}_3$ needs to be linearly independent: $c_1\vec{v}_1 + c_2\vec{v}_2 + c_2\vec{v}_3 = \vec{0}$ only if $c_1 = c_2 = c_3 = 0$, which we have shown in the previous example.

* <u>Basis</u>

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ be some vectors in vector space V.

The vectors are a basis of V if:

1. The vectors must be <u>linearly independent</u>; $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$ only if $c_1 = c_2 = \dots = c_k = 0$.
2. the vectors must span V;
   $\text{Span}\{\vec{v}_1, \vec{v}_2,$